UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 9 to 12

PAPER 44

GALAXIES

Attempt ***THREE*** *questions*

There are ***four*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Show that the gravitational potential energy of a spherical system can be written

$$W = -\frac{G}{2}\int_0^\infty \frac{M^2(r)dr}{r^2},$$

where $M(r)$ is the mass interior to radius r.

(b) Derive the virial theorem for a self-gravitating system of N identical points, each of mass m. Define the gravitational radius R_g in terms of W by the formula:

$$W \equiv -G\,(mN)^2\,/R_g\,.$$

Find the rms velocity dispersion $\langle v_{eq}^2\rangle$ for a system in equilibrium [the answer to be phrased in terms of $(N,\, m,\, R_g)$].

(c) Find the rms escape velocity $\langle v_{esc}^2\rangle$ in the same units and also in terms of $\langle v_{eq}^2\rangle$.

(d) If a system is not in equilibrium, but collapses from being at rest at infinity and is observed in free fall at a time when its gravitational energy is W (defined above), what its velocity dispersion $\langle v_{coll}^2\rangle$ in units of $\langle v_{eq}^2\rangle$?

2 (a) A popular fit to the radial density distribution resultant from dark matter simulations of halos is the "NFW profile":

$$\rho_D(r) \equiv \quad \rho_1\,\frac{r_1^3}{r(r+r_1)^2}\,.$$

Derive the mass distribution for this density law and find the circular velocity $v_{NFW,c}\,(r)$. Make a rough sketch of this function indicating the asymptotic ($r \to 0$, $r \to \infty$) dependencies. Estimate the maximum rotation velocity $v_{NFW,\,c,\,max}$ in units of (G, ρ_1, r_1) , and the radius at which this maximum occurs.

(b) Does the above derived functional form for $v_c(r)$ agree with observed galaxy rotation curves? If not, how is the difference to be understood?

(c) Assume that a galaxy initially has an NFW profile as above $\rho_D(r)$ and slowly accumulates additional mass until the resulting profile is flat with

$$v_c' \;=\; \text{const} \;=\; 2\; v_{NFW,\,c,\,max}\,.$$

After the other matter $\rho_B(r)$ has settled down and the dark matter has reached a new equilibrium $\rho_D'(r)$ find expressions for the ratios:

$$\frac{\rho_D'(r)}{\rho_D(r)} = f(r)\,; \qquad \frac{\rho_D'(r)}{\rho_B(r)} = g(r)\,;$$

in the domain $r \ll r_1$. [Hint: recall "adiabatic invariants".]

3 We may study the vertical structure of a thin axisymmetric disk by neglecting all radial derivatives and adopting the form $f = f(E_z)$ for the distribution function, where $E_z \equiv \frac{1}{2}v_z^2 + \Phi(z)$. Show that if $f = \rho_0(2\pi\sigma_z^2)^{-1/2}exp(-E_z/\sigma_z^2)$, the approximate form of Poisson's equation may be written:

$$2\frac{d^2\phi}{d\zeta^2} = e^{-\phi}, \quad \text{where } \phi \equiv \frac{\Phi}{\sigma_z^2}, \quad \zeta \equiv \frac{z}{z_0} \quad \text{and} \quad z_0 \equiv \frac{\sigma_z}{\sqrt{8\pi G\rho_0}} .$$

By solving this equation subject to the boundary conditions

$$\phi(0) = d\phi/d\zeta|_0 = 0\,,$$

show that the density ρ in the disk is given by

$$\rho(z) = \rho_0 \,\mathrm{sech}^2\left(\frac{1}{2}z/z_0\right) .$$

4 There is a massive black hole (MBH) of mass M_H at the centre of a system of N stars each of mass m. In the vicinity of the black hole, gravitational collisions amongst the stars scatter them into a loss cone (in velocity space) whence, they plunge directly to the Schwarzschild radius of the MBH,

$$R_{sch} \;\equiv\; 2GM_H/c^2\,,$$

and are eaten whole.

(a) Estimate the angular size of this loss cone in velocity space as a function of r.

(b) Estimate the rate at which the MBH eats stars.

(c) Estimate the rate at which the remaining cluster contracts or expands.