UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2002 9 to 12

PAPER 42

STRUCTURE AND EVOLUTION OF STARS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

Substantially complete questions are strongly preferred to fragments

The notation used is standard and the equations of stellar structure are in the form:

$$\frac{dP}{dr} = -\frac{\mathrm{G}m\rho}{r^2} \; ; \; \frac{dm}{dr} = 4\pi r^2 \rho;$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi a c r^2 T^3} \; ; \; \frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon;$$

$$P = \frac{\Re \rho T}{\mu} + \frac{1}{3} a T^4.$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe what is meant by a polytrope with polytropic index n and deduce that the dimensionless variables ξ and θ, where the radius $r = \alpha\xi$, with $\alpha = \text{const.}$, and the density ρ is related to the central density ρ_c by $\rho = \rho_c\theta^n$, satisfy the Lane–Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n .$$

What are the appropriate boundary conditions when the polytrope represents a star?

Deduce further that the mass at radius r is given by

$$m = -4\pi\rho_c\alpha^3\xi^2\frac{d\theta}{d\xi} .$$

Show that when $n = 5$ there is a solution to the Lane–Emden equation of the form

$$\theta = \left(1 + \beta\xi^2\right)^{\gamma} ,$$

where β and γ are constants which you should identify. Find the mass and mean density of the star in this case.

Suppose such a polytrope represents the core of a star burning hydrogen by the CNO-cycle in which the central temperature is T_c and the energy generation rate is given by $\epsilon = \epsilon_0\rho T^{11}$. Deduce that the total luminosity of the burning core is

$$L = A\epsilon_0\alpha^2\rho_c^2 T_c^{11} ,$$

and estimate the order of magnitude of the numerical constant A.

Why is this unlikely to be a good model in practice?

[You may use the fact that $\int_0^1 u^2(1-u^2)^m\, du = m!(m+1)!2^{2m+1}/(2m+3)!$ *when* m *is an integer.]*

2 Briefly describe why, for a homologous set of stars of radius R and mass M we expect $P_\mathrm{c} \propto M\rho_\mathrm{c}/R$ where P_c and ρ_c are the central pressure and density.

A homologous set of fully radiative stars of uniform composition have opacity obeying Kramers' Law $\kappa = \kappa_0 \rho T^{-3.5}$ and generate energy via the *pp*-chain at a rate given by $\epsilon = \epsilon_0 \rho T^{3.5}$, where κ_0 and ϵ_0 are constants and ρ and T are density and temperature which are related to pressure P by an ideal gas equation of state. Show that the luminosity L and radius R of these stars depend on mass M according to

$$L \propto M^{11/2} \qquad \text{and} \qquad R = \text{const.}$$

Explain why such a set of stars is a useful representation of the lower main sequence including the Sun.

Given that the Sun has a central temperature of $1.5 \times 10^7\,\mathrm{K}$ and that the CNO-cycle dominates the *pp*-chain above $2 \times 10^7\,\mathrm{K}$ in solar composition material, estimate the maximum mass represented by the above homologous series.

Above this mass the energy generation rate obeys $\epsilon = \epsilon_0' \rho T^{11.5}$. In all other respects the stars are similar to the homologous series described above. Obtain similar relations between L and R, and M for such stars.

Sketch the two sequences in the H-R diagram.

How does the energy generation rate depend on composition for the *pp*-chain and the CNO-cycle?

Another set of stars, formed in a different environment, have the same mass fraction of hydrogen but a mass fraction of CNO elements 256 times less than the Sun. Assuming any changes in opacity and mean molecular weight are negligible, estimate the temperature at which the CNO-cycle dominates in these stars and the corresponding mass at which this occurs.

Sketch, on the same H-R diagram as the solar composition stars, the two sequences for these stars.

3 A galaxy is 10 Gyr old and has been forming stars at a constant rate since its birth. Show that the fraction Y of stars younger than t is

$$Y(t) = 0.1\left(\frac{t}{\mathrm{Gyr}}\right).$$

Stars in this galaxy form according to an initial mass function

$$n(M)\,dM = \begin{cases} k\left(\frac{M_\odot}{M}\right)^3 dM & M > 0.2\,M_\odot, \\ 0 & M < 0.2\,M_\odot \end{cases},$$

where $n(M)\,dM$ is the number of stars with masses between M and $M + dM$ and k is a constant. Show that the fraction X of stars with mass greater than M is

$$X(M) = \frac{0.04}{(M/M_\odot)^2} \qquad M > 0.2\,M_\odot.$$

A star of mass M spends $8\,\mathrm{Gyr}/(M/M_\odot)^2$ on the main sequence and then a further $2\,\mathrm{Gyr}/(M/M_\odot)^2$ as a red giant before becoming a white dwarf.

On a sketch of the (M, t) plane indicate the area that contains stars which are currently red giants. Show that this maps on to a triangle in the (X, Y) plane and hence show that 0.5% of stars are currently giants and that 2% are currently white dwarfs.

All stars in this galaxy actually form in binary systems with the masses of both components drawn independently from the above distribution. Assume that the components evolve independently. By considering subdivisions of a cube or otherwise show that:

(i) just under 1% of systems contain a red giant;

(ii) just over 3% of systems containing a red giant also contain another evolved star (i.e. another red giant or a white dwarf);

(iii) for every binary containing two red giants there are eight containing a red giant and a white dwarf and sixteen containing two white dwarfs.

[The volume of a pyramid is one third the area of its base times its height.]

4 Write an essay on the origin of the elements. Include in your discussion (a) the various phases of nuclear burning in stars of 1, 5 and $20\,M_\odot$, (b) how nucleosynthetic products are returned to the interstellar medium, (c) how and where elements with atomic masses greater than that of iron might be formed and (d) a brief discussion of the importance of binary stars.