UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 1.30 to 4.30

PAPER 41

ASTROPHYSICAL FLUID DYNAMICS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

Candidates may bring their notebooks into the examination

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The expansion factor $R(t)$ of a homogeneous flat Newtonian universe of density $\rho(t)$ and pressure $p(t)$, with $p = K\rho^\gamma$, satisfies $\dot{R} = 2(9t_0^2 R)^{-1/2}$, with $R(t_0) = 1$, where K and the adiabatic exponent γ are constants and $4 < 3\gamma < 5$. Consider the universe to suffer a small adiabatic density perturbation $\rho'(\boldsymbol{x}, t) = \Re\{\varepsilon(t)\rho(t)\exp(iR^{-1}\boldsymbol{k}.\boldsymbol{x})\}$. From the differential satisfied by ε, establish the equation

$$\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2} + \frac{4}{3t}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \left(\frac{\Lambda^2}{t^{2\gamma-2/3}} - \frac{2}{3t^2}\right)\varepsilon = 0,$$

where $\Lambda = kt_0^{\gamma-1/3}c(t_0)$, $\;k = |\boldsymbol{k}|$ and $c(t)$ is the adiabatic sound speed. Hence show that

$$\varepsilon = t^{-1/6}\left[AJ_{-\lambda}\left(\frac{\Lambda}{\nu t^\nu}\right) + BJ_\lambda\left(\frac{\Lambda}{\nu t^\nu}\right)\right],$$

where A and B are constants, depending on the initial conditions, and λ and ν are constants, depending on γ, whose values you should determine.

Demonstrate that if t is small, the perturbation oscillates with an amplitude that decreases with time, but that if t is large the perturbation grows in proportion to $t^{2/3}$. Show that the condition for growth is that

$$\frac{ck}{R} \lesssim \sqrt{6\pi\nu G\rho}.$$

Interpret this condition.

2 A stellar cluster is composed of a central massive object of mass $M_{\rm c}$ and a number of lighter stars which may be considered to be moving with random velocities v such that statistically their kinetic energies are in balance with their gravitational potential energies: $\frac{1}{2}v^2 \simeq GM_{\rm c}/R$, where R is distance from the centre of the cluster. The cluster is accreting gas (formally from being at rest at infinite distance from the cluster) at a constant rate A.

Argue that if for the purpose of computing the velocity of infalling gas in the cluster when it is far from any particular star ('far' is where the gravitational pull of that particular star is much less than that of the central massive object) the total gravitational potential of all the stars can be ignored compared with that of the central massive object, then the typical relative velocity V of the gas relative to a typical star (when it is far from that star) is proportional to $R^{-1/2}$. How does the gas density ρ vary with R?

Each star accretes some of the gas at a rate determined by the Bondi-Hoyle formula $\dot{M} = 4\pi G^2 M^2 V^{-3} \rho$, where M is the mass of the star. The remaining gas is accreted by the central object, whose increase of mass may be neglected. Establish the equation

$$\dot{M} = \Lambda M^{\alpha} R^{\beta},$$

where Λ, α and β are constants. Hence obtain

$$M = \frac{M_0}{1 - \lambda M_0 t},$$

where M_0 is the initial value of M and λ is independent of time t.

The number of stars $\mathrm{d}N = F(M)\mathrm{d}M$ with masses between M and $M + \mathrm{d}M$ at time t is evidently the same as $F_0(M_0)\mathrm{d}M_0$, the number of stars with masses between M_0 and $M_0 + \mathrm{d}M_0$ at $t = 0$. Deduce that

$$F(M) = \left(\frac{M_0}{M}\right)^2 F_0\left(\frac{M}{1 + \lambda M t}\right),$$

and deduce that if there is initially a small range of stellar masses the asymptotic limit of the mass spectrum approximately satisfies

$$\mathrm{d}N \propto M^{-2}\mathrm{d}M$$

over a wider range of M.

Discuss the limitations of the assumptions adopted to obtain this result.

3 The outer layers of a star may be approximated as being plane parallel, in hydrostatic support under constant gravitational accelaration g. Consider the star to undergo small-amplitude adiabatic oscillations proportional to $\mathrm{e}^{i(\boldsymbol{k}.\boldsymbol{x}-\omega t)}$. Write down the differential equations governing the vertical component $\xi(z)$ of the displacement amplitude and the Lagrangian pressure perturbation $\delta p(z)$, in the Cowling approximation, where $\boldsymbol{k}$ is a constant horizontal wavenumber and z is measured downwards. Confirm that the equations admit the f-mode solution with frequency satisfying $\omega^2 = gk$, where $k = |\boldsymbol{k}|$. Write down the form of eigenfunctions in this case.

The displacement amplitude of the pure f mode diverges as $z \to -\infty$; consequently, whatever the amplitude of the mode, it must be nonlinear high in the atmosphere and cannot sustain its pure linear form. However, there is an interfacial mode which resembles the f mode deep in a (late-type) star but deviates from it in and above the chromosphere, where temperature (and hence sound speed c) increases rapidly, and consequently density decreases, with height. For that mode $\xi \to 0$ as $z \to -\infty$. In order to obtain an approximation to this mode it is useful to consider two separate expansions above and below some fiducial level $z = z_0$ situated immediately above the chromosphere:

For $z > z_0$, the mode resembles the pure f mode. Accordingly, write

$$\xi = \mathrm{e}^{-k(z-z_0+\varepsilon\phi)}, \quad \delta p = \varepsilon f \xi, \quad \omega^2/gk = 1-\varepsilon,$$

where ε is a small parameter which characterizes the ratio of the densities ρ above and below the chromosphere, and where $\phi(z)$ and $f(z)$ may be considered, if you like, to be expanded in powers of ε. It is convenient to insist that $\phi(z_0) = 0$. Why has δp been taken to be $\mathrm{O}(\varepsilon)$? Write down the differential equations satisfied by ϕ and f, to leading order in ε, and hence show that

$$f(z) \simeq -2gk\mathrm{e}^{2k(z-z_0)} \int_z^\infty \mathrm{e}^{-2k(z'-z_0)}\rho(z')\,\mathrm{d}z'.$$

For $z < z_0$, where ρ is small, set $\rho = \varepsilon\hat{\rho}$ and $c^2 = \varepsilon^{-1}\hat{c}^2$, and write

$$\delta p = \varepsilon A \mathrm{e}^{k(z-z_0+\varepsilon\phi)} \equiv \varepsilon\psi, \quad \xi = \mu\psi,$$

in which ϕ and μ may be considered to be expanded in powers of ε. Once again, obtain an approximation, to leading order in ε, to the solution, and hence, by applying continuity conditions at $z = z_0$, establish that

$$\varepsilon^{-1} \simeq 2k^2 \int_{-\infty}^{z_0} \mathrm{e}^{2k(z-z_0)}\rho^{-1}(z)\mathrm{d}z \int_{z_0}^\infty \mathrm{e}^{-2k(z-z_0)}\rho(z)\,\mathrm{d}z.$$

State any conditions you consider should be satisfied in order for this approximation to be valid.

4 The equation of conservation of momentum governing spherically symmetrical motion of a star may be written

$$\frac{\partial p}{\partial m} + \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{4\pi r^4},$$

where m is the mass enclosed in the sphere of radius r and the time derivative is taken at constant m. Show that the linearized equation relating the Lagrangian pressure perturbation $\delta p(r)\mathrm{e}^{-i\omega t}$ to the displacement $\delta r = r\xi(r)\mathrm{e}^{-i\omega t}$ may be written

$$\frac{\mathrm{d}\delta p}{\mathrm{d}m} - \frac{\omega^2\xi}{4\pi r} - \frac{Gm\xi}{\pi r^4} = 0,$$

and that the Lagrangian density perturbation $\delta\rho(r)\mathrm{e}^{-i\omega t}$ satisfies

$$\delta\rho = -\rho\left(r\frac{\mathrm{d}\xi}{\mathrm{d}r} + 3\xi\right) \equiv -\rho\chi.$$

Hence show that adiabatic perturbations satisfy

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\gamma p\left(r\frac{\mathrm{d}\xi}{\mathrm{d}r} + 3\xi\right)\right] - 4\frac{\mathrm{d}p}{\mathrm{d}r}\xi + \omega^2\rho r\xi = 0. \qquad (*)$$

This is an eigenvalue equation for ω^2.

Cast the equation into self-adjoint form: $\frac{\mathrm{d}}{\mathrm{d}r}[f(r)\frac{\mathrm{d}\xi}{\mathrm{d}r}] + g(r)\xi = 0$, and show that if $p = 0$ at $r = R$, the eigenfrequency ω satisfies $\omega^2 = K/I$, where

$$K = \int_0^R \left\{\gamma p r^4\left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)^2 - r^3\frac{\mathrm{d}}{\mathrm{d}r}[(3\gamma-4)p]\xi^2\right\}\mathrm{d}r$$

and

$$I = \int_0^R r^4\rho\xi^2\,\mathrm{d}r.$$

In view of the symmetry of the integrals, the ratio K/I provides a variational principle for ω^2 amongst bounded twice-differentiable functions ξ. In particular, the lowest eigenvalue ω^2 is a minimum of K/I.

Integrate the second term in the integrand for K by parts, and express ξ in terms of χ and $\mathrm{d}\xi/\mathrm{d}r$ to show that

$$K = \frac{1}{3}\int_0^R \left[4r^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)^2 + (3\gamma-4)\chi^2\right]pr^2\,\mathrm{d}r.$$

Deduce that a necessary condition for the star to be dynamically unstable is that $\gamma < \frac{4}{3}$ somewhere.

It can be shown that the bounded eigenfunctions of equation (*) span the space of bounded twice-differentiable functions. By considering the value of the functional $K(\xi)$ when $\xi =$ constant, or otherwise, show that a necessary condition for dynamical stability is that $\gamma > \frac{4}{3}$ somewhere.

Comment on the usefulness of these conditions for determining the stability of a star.

[*Bessel's equation is*

$$\frac{\mathrm{d}^2 J_\nu}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}J_\nu}{\mathrm{d}z} + (1 - \frac{\nu^2}{z^2})J_\nu = 0.$$

If $|z|$ *is small compared with unity,* $J_\nu \simeq (\frac{1}{2}z)^\nu$*; if* $|z|$ *is substantially greater than unity,* $J_\nu \simeq (\frac{2}{\pi z})^{\frac{1}{2}} \cos(z - \frac{1}{2}(\nu + \frac{1}{2}\pi))$.]