UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2002 9 to 12

PAPER 4

PARTIALLY ORDERED GROUPS

*Attempt **FOUR** questions*

*There are **six** questions in total*

The questions carry equal weight

NOTATION: *In the questions, standard notation has been used: o-group is a totally ordered group, ℓ-group is a lattice-ordered group, ℓ-homomorphism is a group and lattice homomorphism, ℓ-subgroup is a sublattice subgroup, and prime subgroups are always convex ℓ-subgroups. Further, if G is an ℓ-group, then $G_+ = \{g \in G : g > 1\}$; $\mathbb{R}$ stands for the additive group of real numbers with the usual ordering and $\mathbb{Z}_+$ denotes the set of all strictly positive integers.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 **(i)** Let P be a convex ℓ-subgroup of an ℓ-group G. Prove that its set of right cosets is partially ordered by: $Pg \leqslant Ph$ iff $g \leqslant ph$ for some $p \in P$. Prove that this partial order is total iff (for any convex ℓ-subgroups C, D, if $C \cap D = P$, then $C = P$ or $D = P$).

(ii) Prove that a lattice-ordered group G has all its values normal iff $|f||g| \leqslant |g|^2|f|^2$ for all $f, g \in G$. Deduce that every o-group is normal-valued.

2 State and prove Ohnishi's condition for a group to have an order with respect to which it is an o-group. Deduce that a group can be made into an o-group iff all its finitely generated subgroups can. Hence show that every torsion-free Abelian group is an o-group for some order.

3 (A) Prove that every ℓ-group is ℓ-isomorphic to a group of order-preserving permutations (with the pointwise ordering) of some totally ordered set.

(B) Prove that every 2-transitive ℓ-permutation group is n-transitive for all positive integers n.

(C) State the Trichotomy classification of primitive transitive ℓ-perm- utation groups. Which primitive transitive ℓ-permutation groups satisfy $|f||g| \leqslant |g|^2|f|^2$ for all f, g. Justify your answer.

4 Let $G = Aut(\mathbb{R}, \leqslant)$.

(i) Let $f_1, f_2 \in G_+$ such that f_1, f_2 have supports that are bounded open intervals in $\mathbb{R}$. Prove that f_1 and f_2 are conjugate in G.

(ii) Now let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma < \alpha < \beta$. Let $a, f, g, h \in G$ with $f, g \in G_+$ each comprising one bounded bump and $\alpha < supp(a) < \beta < \alpha f, \alpha g$. Suppose further that $\gamma < supp(f) \cup supp(g) < \gamma h$. Prove that there is $b \in G$, commuting with a and h such that $b^{-1}fb = g$.

5 Let G be an Abelian o-group that is also a vector space over $\mathbb{R}$. Suppose that the set of convex subgroups of G (other than G) is precisely $\{C_n : n \in \mathbb{Z}_+\}$ where (i) $C_1 = \{0\}$, (ii) $\bigcup\{C_n : n \in \mathbb{Z}_+\} = G$, (iii) $C_n \subseteq C_m$ whenever $n \leqslant m$, and (iv) $C_{n+1}/C_n \simeq \mathbb{R}$ for all $n \in \mathbb{Z}_+$.

Prove directly that G can be ℓ-embedded in the Hahn group $V(\mathbb{Z}_+, \mathbb{R})$.

[Hint: The lemma used in the proof of the general Hahn Theorem can be established directly under the hypotheses (i) — (iv).]

6 **(i)** Can a two generator Abelian o-group G have more than three convex subgroups? Explain.

(ii) Describe the free Abelian ℓ-group on two generators and outline the construction. Is it isomorphic as a group to $\mathbb{Z}^n$ for some $n \in \mathbb{Z}_+$? Explain.

(iii) Is the additive group of all real-valued functions from the half plane in $\mathbb{R}^2$ defined by $x \leqslant y$ ℓ-isomorphic to the free Abelian ℓ-group on two generators? Explain (quoting explicitly any theorems that you use).

(iv) Must an ℓ-homomorphic image of the free Abelian ℓ-group on two generators be Archimedean? Explain.