UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 31 May 2002 1.30 to 3.30

# PAPER 32

# MATHEMATICAL MODELS IN FINANCIAL MANAGEMENT

*Attempt* ***THREE*** *questions*

*There are* ***five*** *questions in total*

*The questions carry equal weight*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1 (a)** In the returns matrix below each column represents the state-contingent return of an asset. Is this matrix arbitrage free for the three assets shown? If not, try to find arbitrage opportunities where you commit zero capital now, and arbitrage opportunities where your position yields no future inflows or outflows of capital.

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

**(b)** (i) The supporting price vector in a four state economy with a complete market is given by

$$p = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.3 \end{pmatrix}$$

If a risk-free asset exists, what is its return? Use the risk-free return to derive the risk-neutral probability measure.

(ii) An agent has a von Neumann-Morgenstern utility function $u$ defined over future state-contingent consumption $\tilde{c}$

$$u(\tilde{c}) := \ln(\tilde{c}) .$$

Use the supporting pricing vector in (i) to compute the optimal consumption plan for this agent when the states are equally likely and he has 100 currently available for investment.

**(c)** (i) Now suppose asset returns have the form

$$\tilde{r}_i = a_i + b_i \tilde{f}_1 + c_i \tilde{f}_2 ,$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are independent random variables.

If the return on the risk-free asset is 1.05, the expected return on an asset $i$ with $b_i = 1$ and $c_i = 0$ is 1.10 and the expected return on an asset $j$ with $b_j = 0$ and $c_j = 1$ is 1.15, what is the expected return on an asset $k$ with $b_k = \beta$ and $c_k = \gamma$?

(ii) Suppose there is a risk-free asset with return $R$. In the absence of arbitrage, for any three risky assets (linearly independent) the following matrix must be singular:

$$\begin{pmatrix} a_1 - R & a_2 - R & a_3 - R \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} .$$

Explain.

(iii) Use the fact that the matrix in (ii) is singular to derive the arbitrage-pricing theory for the special case where assets have no idiosyncratic risk.

**2** The portfolio mean-variance efficient frontier for an economy with a risk free asset with return $R$ is given by

$$w = \lambda \Sigma^{-1}(z - R.1),$$

where $w$ is the vector of portfolio weights across all risk assets, $\Sigma$ is the variance- covariance matrix and $z$ is the vector of expected returns for the risky assets. The expression $R.1$ is the product of a scalar ($R$) and a vector of ones (1). The parameter $\lambda$ is a Lagrange multiplier for the programme

$$\min \frac{1}{2} w' \Sigma w$$

subject to

$$w'(z - R.1) = \mu - R,$$

where $\mu - R$ is the expected excess return on the portfolio $w$.

**(a)** Find an expression for the Lagrange multiplier $\lambda$.

**(b)** Find the expected returns on the efficient frontier (i.e. the correspondence between the variance and the expected return of the efficient portfolios).

**(c)** Find the expected return and the variance of the tangent portfolio (i.e. the efficient portfolio with zero weight on the risk free asset).

**3 (a)** Show that American calls on non-dividend paying stocks are never exercised before maturity. Argue that, on the other hand, it would not be rational never to exercise an American put before maturity.

**(b)** Consider an American call on a stock with price $\mathbf{S}$ of which you are certain that a dividend of $D$ will be paid at date $t_1 > t$. Argue that the call will not be exercised before $t_1$ and that exercise will not occur before final maturity date $T$ if

$$D < (1 - B(\tau_1))X,$$

where $\tau_1 = T - t_1$ is time-to-maturity at $t_1$, $B(\tau_1)$ is the price at $t_1$ of a pure discount bond maturing at $T$ and $X$ is the call's exercise price.

**(c)** State and prove the *put-call parity* relationship between a European put and call with identical exercise prices and times-to-maturity, written on a stock without dividends. Does this relation hold for American puts and calls? Explain.

**(d)** A *European binary call* with strike price $X$ and exercise date $T$ written on a stock price $\mathbf{S}$ gives the right to a payoff of 1 if $S_T \geqslant X$ and 0 otherwise. Likewise, a *European binary put* will give a payoff of 1 at $T$ if $S_T < X$, and 0 otherwise. Letting $C_t^{bin}$ and $P_t^{bin}$ denote the price at $t < T$ of the binary call and put, respectively, derive a put-call type relationship between $C_t^{bin}$ and $P_t^{bin}$ .

**4** Consider an infinite maturity put option written on a stock price that follows the risk adjusted process

$$d\mathbf{S}_t = (r - \delta)\mathbf{S}_t dt + \sigma \mathbf{S}_t d\mathbf{W}_t.$$

**(a)** For a given exercise trigger $\underline{S}$ price an infinite maturity put option with a strike price $X$ written on the above security price (considering Merton's version of the Black-Scholes PDE).

**(b)** Derive the optimal exercise price $\underline{S}^*$ for the put option.

**(c)** Show how the trigger price is affected by changes in $\delta$.

**5** Write a short essay on credit derivatives. Your answer should include a description of the basic products and their current market, the uses of credit derivatives for both risk management and investment and a discussion of their pricing.