UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 1.30 to 3.30

## PAPER 28

## INTERACTING PARTICLE SYSTEMS

*Attempt* ***THREE*** *questions*

*There are* ***four*** *questions in total*

*The questions carry equal weight*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Describe the bond percolation model with parameter $p$ on the square lattice $\mathbb{Z}^2$. What does it mean to say that an event associated with this process is *increasing*?

State the (Harris)–FKG inequality and the disjoint-occurrence inequality for two increasing events.

Let $\Lambda_n = [-n,n]^2$ and $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$, and let $g_k = P_p(0 \leftrightarrow \partial\Lambda_k)$ be the probability of an open path in the process joining the origin to some vertex of $\partial\Lambda_k$. Show that

$$g_n \leqslant g_{n-m} \sum_{x \in \partial\Lambda_m} P_p(0 \leftrightarrow x), \quad m \leqslant n.$$

Let $\chi(p)$ be the mean size of the open cluster containing the origin, and assume that $p$ is such that $\chi(p) < \infty$. Show that there exists $\gamma > 0$ such that $g_k \leqslant e^{-\gamma k}$ for all $k$.

**2** Let $T_d$ be a homogeneous infinite tree in which each vertex has degree $d+1$. Let $\xi_t^A$ be the set of infected vertices at time $t$ in the contact model on $T_d$ with infection rate $\lambda$ and death rate 1, under the assumption that the infected set at time 0 is the non-empty finite set $A$. The corresponding probability measure is written $P_\lambda$, with expectation $E_\lambda$.

Let $0 < \rho < 1$ and define $\nu_\rho(B) = \rho^{|B|}$, for a set $B$ of vertices. Show that

$$\frac{d}{dt} E_\lambda\left(\nu_\rho\left(\xi_t^A\right)\right)\Big|_{t=0} \leqslant (1-\rho)\nu_\rho(A)\left[\frac{|A|}{\rho}(1-\lambda\rho(d-1)) - 2\lambda\right].$$

Deduce that $E_\lambda(\nu_\rho(\xi_t^A))$ is non-increasing in $t$, if $\rho\lambda(d-1) \geqslant 1$.

Let $\lambda_1 = \inf\{\lambda : \ P_\lambda(\xi_t^{\{x\}} \neq \phi \text{ for all } t) > 0\}$, where $x$ denotes a vertex of the tree. Show that $\lambda_1 < 1/(d-1)$.

To each vertex $x$ of $T_d$ is allocated an integer $g(x)$ in such a way that: if $x, y$ are neighbours, then $g(y) = g(x) \pm 1$, and furthermore each $x$ has exactly one neighbour $y$ with $g(y) = g(x) - 1$. By considering the function $w_\rho(A) = \sum_{x \in A} \rho^{g(x)}$ or otherwise, where $0 < \rho < 1$ and $A \subseteq V$, show that

$$\lambda_2 = \inf\{\lambda : P_\lambda(x \in \xi_t^{\{x\}} \text{ for unbounded } t) > 0\}$$

satisfies $\lambda_2 \geqslant 1/\{2\sqrt{d}\}$.

**3** Let $G = (V, E)$ be a finite graph and let $0 < p < 1$ and $q \in \{2, 3, \ldots\}$. On the product sample space $\{1, 2, \ldots, q\}^V \times \{0, 1\}^E$ we define the probability mass function

$$\mu(\sigma, \omega) = \frac{1}{Z} \prod_{e \in E} \{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\},$$

for $\sigma \in \{1, 2, \ldots, q\}^V$, $\omega \in \{0, 1\}^E$, where $\delta_{r,s}$ is the Kronecker delta, and $\delta_e(\sigma) = \delta_{\sigma(x),\sigma(y)}$ where $e$ is the edge with endvertices $x$ and $y$. Here $Z$ is a constant depending on $p$ and $q$.

Show that the marginal mass functions $\mu_1(\sigma) = \sum_\omega \mu(\sigma, \omega)$, $\mu_2(\omega) = \sum_\sigma \mu(\sigma, \omega)$ are given by the Potts and random-cluster measures

$$\mu_1(\sigma) = \frac{1}{Z'} \exp\left(\beta \sum_e \delta_e(\sigma)\right), \text{ where } e^{-\beta} = 1 - p,$$

$$\mu_2(\omega) = \frac{1}{Z''} \left\{\prod_e p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega)},$$

for constants $Z', Z''$, where $k(\omega)$ is the number of open clusters under $\omega$.

Find the conditional mass function $\mu(\sigma \,|\, \omega)$ of $\sigma$ given the edge-configuration $\omega$. Deduce that, for $x, y \in V$,

$$\mu_1(\sigma(x) = \sigma(y)) - \frac{1}{q} = (1 - q^{-1})\ \mu_2(x \leftrightarrow y),$$

where $\{x \leftrightarrow y\}$ is the event that $\omega$ contains an open path from $x$ to $y$.

**4** Let $G = (V, E)$ be a finite regular graph (i.e., each vertex has the same number of neighbours). Particles inhabit the vertices in $V$, and each vertex may be occupied by no more than one particle at any time. Each particle jumps at rate 1, and when it jumps it does so to a neighbour chosen uniformly at random. If this neighbour is already occupied by a particle then the jump does not take place. You may assume the maximum amount of independence between jumps.

Let $\eta_t^A$ be the set of vertices occupied by the particles at time $t$, where $A$ is the set of their initial positions. Show that

$$P\big(\eta_t^A \supseteq B\big) = P\big(\eta_t^B \subseteq A\big)$$

for $A, B \subseteq V$.

For $0 \leqslant \rho \leqslant 1$, let $\mu_\rho$ be product measure on the configuration space $\{0, 1\}^V$ with density $\rho$; i.e., each vertex is occupied with probability $\rho$, independently of all other vertices. Show that, for all $0 \leqslant \rho \leqslant 1$, the measure $\mu_\rho$ is invariant for the above exclusion process.