UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2002 1.30 to 3.30

PAPER 25

ANALYTIC NUMBER THEORY

*Attempt **TWO** questions*

*There are **five** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Illustrate the methods of Tchebychev and Mertens by proving that

$$\sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1).$$

Deduce by partial summation, or otherwise, that

$$\sum_{p \leqslant x} \frac{\sqrt{\log p}}{p} = 2\sqrt{\log x} + O(1).$$

2 **Either**:

Prove that $\zeta(s) \neq 0$ on the line $\sigma = 1$.

Write down a relation between

$$\int_0^x \psi(u) du$$

and $\zeta'(s)/\zeta(s)$, where ψ is the Tchebychev function. Describe how it leads to a proof of the prime-number theorem.

Or:

State and prove the functional equation for $\zeta(s)$.

State also the Riemann-von Mangoldt formula. Deduce from the latter that if $\gamma_1, \gamma_2, \ldots$ is the increasing sequence of positive ordinates of the zeros of $\zeta(s)$ then there are $\ll \log T$ elements with $T < \gamma_n \leqslant T + 1$. Hence verify that there exists $c > 0$ such that

$$\gamma_{n+1} - \gamma_n > c/\log n$$

for an infinite sequence of values of n.

3 Show how the explicit formula for $\zeta'(s)/\zeta(s)$ can be used to establish a zero-free region for $\zeta(s)$ of the form $\sigma > 1 - c/\log|t|$ for some $c > 0$.

Show further how the formula leads to the estimate

$$\sum_{\rho} \frac{1}{4 + (t - \gamma)^2} \ll \log|t|$$

where the sum is over all non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. Deduce that the number of zeros ρ with $|t - \gamma| < 1$ is $\ll \log|t|$.

4 Describe the main ideas of the Selberg upper-bound sieve. Show how it leads to the result that if p runs through all the primes such that $p,\ p+2$ are twin primes then $\sum 1/p$ converges.

5 Prove Dirichlet's theorem on primes in arithmetical progressions assuming that $L(1,\chi) \neq 0$ for a real, non-principal character χ.

It is known, from the Siegel-Walfisz theorem, that the number $\pi(x,q,a)$ of primes $p \leqslant x$ in the arithmetical progression $a, a+q, a+2q, \ldots$ with $(a,q) = 1$ satisfies $\pi(x,q,a) \sim (1/\phi(q)) \operatorname{li} x$ as $x \to \infty$. Deduce that

$$\sum_{\substack{p \leqslant x \\ p \equiv a (\operatorname{mod} q)}} \frac{1}{p} \sim \frac{1}{\phi(q)} \log\log x \quad \text{as } x \to \infty.$$