UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 9 to 12

PAPER 23

INFINITE AND FINITE MODEL THEORY

Attempt ***THREE*** *questions from Section A AND* ***THREE*** *questions from Section B*

There are ***five*** *questions in Section A and* ***five*** *questions in Section B*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 (a) State the Compactness Theorem.

(b) Let $\mathcal{A}$ be an L-structure and T' a theory in an expansion $L' = L[X]$ of L. Set $S = \{\phi \in L \cap \forall_1 \,|\, T' \vDash \phi\}$. Show that if $\mathcal{A} \vDash S$, then $\mathcal{A}$ embeds in some $\mathcal{B} \vDash T'$.

(c) Let T be a theory in L, and $\mathcal{C}$ an L-structure. Suppose that $\mathcal{C} \vDash \phi$ for all $\exists_1$ sentences ϕ entailed by T. Show that we can find $\mathcal{A} \vDash T$ such that $\mathcal{A}$ embeds in an elementary extension of $\mathcal{C}$.

(d) Let T be a theory. Show that if $\mathrm{Mod}(T)$ is closed under superstructures then T has a set of $\exists_1$-axioms.

2 (a) Let T be a theory and $\phi(\mathbf{x})$ a formula in L. Suppose that whenever we have $\mathbf{a} \in \mathcal{A}$, with embeddings $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{A} \hookrightarrow \mathcal{C}$, then $\mathcal{B} \vDash \phi(\mathbf{a})$ implies $\mathcal{C} \vDash \phi(\mathbf{a})$. Show that ϕ is equivalent mod T to a quantifier-free formula.

(b) Write down axioms for the theory of the random graph. Show that this theory admits elimination of quantifiers.

3 (a) What is a 1-embedding?

(b) Suppose that $\alpha : \mathcal{A} \to \mathcal{B}$ is a 1-embedding. Show that there exists an embedding $\beta : \mathcal{B} \to \mathcal{C}$ with $\beta\alpha : \mathcal{A} \to \mathcal{C}$ elementary.

(c) Let $\mathcal{A}$ be a structure and T a theory in L. Suppose that $\mathcal{A}$ satisfies every $\forall_2$ consequence of T. Show that $\mathcal{A}$ 1-embeds in a model of T.

(d) Let T be a theory. Show that if $\mathrm{Mod}(T)$ is closed under unions of chains, then T has a set of $\forall_2$ axioms.

4 (a) What does it mean for a theory T locally to omit a set $\Gamma(\mathbf{x})$ of formulae? State the Omitting Types Theorem.

(b) Let T be a complete theory in a countable language. What is an atomic model of T? What is an ω-saturated model of T? Suppose that T has a countable ω-saturated model: show that T has a countable atomic model. Does the converse hold?

(c) Let T be the theory of dense linear orders without endpoints, and with countably many constants c_i satisfying $c_i < c_{i+1}$. Up to isomorphism T has three countable models according as (i) c_i has a limit, (ii) c_i is unbounded and (iii) c_i is bounded but with no limit in the model. Which of these is atomic and which ω-saturated? Briefly justify your answer.

5 Let T be a complete theory in a countable language. Show that the following conditions are equivalent.

(i) T is ω-categorical.

(ii) All the Stone spaces $S_n(T)$ are finite.

(iii) T has a countable model $\mathcal{A}$ with $\mathrm{Aut}(\mathcal{A})$ oligomorphic.

[You may use without proof simple properties of boolean algebras, and basic properties of atomic and countably prime models.]

Section B

6 (a) For a structure $\mathcal{A}$ and tuple $\mathbf{a}$ of elements from its universe, define the neighbourhood of radius r around $\mathbf{a}$ in $\mathcal{A}$.

(b) Use your definition in (a) to state Hanf's locality lemma.

(c) Recall that a graph is 2-colourable if and only if it contains no cycle of odd length. Use Hanf's lemma to show that there is no first-order sentence that defines 2-colourable graphs.

7 (a) State Fagin's characterisation of NP.

(b) Use (a) to show that, if there is a sentence of second-order logic that is not equivalent to an existential sentence then P $\neq$ NP. State clearly any properties of the complexity classes P and NP which you use in your argument.

(c) Show that NP $\neq$ PSPACE also follows from the hypothesis in (b).

8 You may assume that the logic PFP characterises the complexity class PSPACE on ordered structures.

(a) Suppose ϕ is a formula of PFP, R is a relation variable and $\mathcal{O}$ is the class of structures interpreting the symbol $<$ as a linear order. Show that there is a formula of PFP equivalent to $\exists R.\phi$ on all structures in $\mathcal{O}$.

(b) Use (a) to deduce that a class $\mathcal{K}$ of structures is definable by a sentence of the form $\exists R.\phi$ (where ϕ is in PFP) if and only if the collection of strings encoding structures in $\mathcal{K}$ is decidable in PSPACE.

9 We say that two formulae $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$, whose free variables are among $\mathbf{x}$, are *almost everywhere equivalent* if the sentence $\forall\mathbf{x}.(\phi \longleftrightarrow \psi)$ has asymptotic probability 1.

(a) What are the extension axioms in the language of one binary relation?

(b) Assuming that each extension axiom has asymptotic probability 1, show that any formula in the language of one binary relation is almost everywhere equivalent to a quantifier-free formula.

10(a) Show that for any positive integer n, there is a sentence λ_n of L^2 such that a linear order is a model of λ_n if and only if it has length at least n.

(b) If $\phi(x)$ is a formula of L^k, $k \geqslant 2$, show that there is a sentence ϕ_n of L^k such that, for any linear order $\mathcal{L}$, $\mathcal{L} \vDash \phi_n$ if and only if there are at least n distinct elements $a \in \mathcal{L}$ such that $\mathcal{L} \vDash \phi[a]$.

(c) Show that in the language with two binary relation symbols, $<$, E, there is, for each n, a sentence η_n of L^3, such that $\mathcal{A} \vDash \eta_n$ if and only if $<$ is a linear order and E is an equivalence relation with at least n equivalence classes.