UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2002 1.30 to 4.30

PAPER 22

ELEMENTARY TOPOSES

Attempt ***THREE*** *questions of which at most* ***TWO*** *should be from Section A*

There are ***four*** *questions in Section A and* ***three*** *questions in Section B*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 Explain what is meant by a regular category, and prove that in a regular category the class of covers (that is, the class of extremal epimorphisms) coincides with the class of regular epimorphisms. Give an example of a category with finite limits in which not every extremal epimorphism is regular.

2 Prove carefully that, for any small category $\mathcal{C}$, the functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos. Show also that if $\mathcal{C}$ has the property that the slice category $\mathcal{C}/A$ is essentially small (i.e., equivalent to a small category) for every object A, then $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is still a topos. Give an example of a category satisfying the latter hypothesis which is not equivalent to a small category.

3 Let $\mathcal{E}$ be a topos. Prove that the functor $\Omega^{(-)} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is monadic. Deduce that any logical functor between toposes preserves finite colimits. Give, with justification, two examples of functors between toposes which have left adjoints, and of which the first preserves exponentials and the second preserves the subobject classifier, but neither of which preserves all finite colimits.

4 Define the notions of *localic* and *hyperconnected* geometric morphisms, and show that every geometric morphism can be factored, uniquely up to equivalence, as a hyperconnected morphism followed by a localic one. Show also that if a geometric morphism can be factored as an inclusion followed by a hyperconnected morphism, then it has an alternative factorization as a hyperconnected morphism followed by an inclusion.

Section B

5 Let $\mathcal{E}$ be a category with finite limits and a subobject classifier Ω. Suppose given a monomorphism $f\colon \Omega \rightarrowtail \Omega$ in $\mathcal{E}$; by considering the subobject $g\colon U \rightarrowtail \Omega$ classified by f, and the subobject $V \rightarrowtail U$ classified by g, show that $ffg = g$, and deduce that $f^2 = 1_\Omega$.

By considering the topos $[\mathbf{N}, \mathbf{Set}]$ where $\mathbf{N}$ is the ordered set of natural numbers, or otherwise, show that the subobject classifier in a topos may have epic endomorphisms which are not isomorphisms.

6 Let $\mathcal{E}$ be a topos, and let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{E}$ which is cartesian and idempotent (i.e. such that T preserves finite limits and μ is an isomorphism). Show that the category of $\mathbb{T}$-algebras is equivalent to the category of sheaves for a suitable local operator on $\mathcal{E}$, and deduce that it is a topos. [*You should describe how the topos structure is defined on the category of sheaves, but detailed verification that the constructions work is not required.*]

Let $T\colon \mathbf{Set} \to \mathbf{Set}$ be the functor $A \mapsto A^2$. Show how to define a monad structure $\mathbb{T}$ on T, in such a way that specifying a $\mathbb{T}$-algebra structure on a nonempty set A is equivalent to specifying a product decomposition $A \cong B \times C$. Is the category of $\mathbb{T}$-algebras cartesian closed? Is it a topos?

7 Let $\mathcal{C}$ be a small category satisfying the following three conditions:

(a) The objects of $\mathcal{C}$ are the natural numbers, and $\mathcal{C}\,(m, n) \neq \emptyset$ iff $m \geqslant n$.

(b) If f, g, h, k are four morphisms of $\mathcal{C}$ such that $fh = gk$, and additionally the domains (as well as the codomains) of f and g are equal, then $f = g$.

(c) Given $f\colon m \to n$ and $g\colon m \to (n+1)$ in $\mathcal{C}$, there exist $h, k\colon (m+1) \rightrightarrows m$ such that $fh = fk$ but $gh \neq gk$.

We say that a sieve R on an object n of $\mathcal{C}$ is *persistently nonempty* if, for every morphism f with codomain n, there exists a member of R which factors through f. Show that

(i) The family of all persistently nonempty sieves on objects of $\mathcal{C}$ forms a (Grothendieck) coverage J on $\mathcal{C}$;

(ii) The topos of J-sheaves is Boolean and two-valued (i.e. its terminal object has precisely two subobjects);

(iii) Each representable functor $\mathcal{C}\,(-, n)$ is a subobject of its associated J-sheaf (A_n, say);

(iv) There are no morphisms $A_n \to A_{n+1}$ in $\mathbf{Sh}(\mathcal{C}, J)$ [*hint: show that any partial map* $\mathcal{C}\,(-, n) \rightharpoonup \mathcal{C}\,(-, n+1)$ *in* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ *has empty domain*];

(v) Each A_n maps epimorphically to 1 in $\mathbf{Sh}(\mathcal{C}, J)$, but $\prod_{n\in\mathbb{N}} A_n \cong 0$.