UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 31 May 2002 9 to 12

PAPER 20

CATEGORY THEORY

*Attempt **SIX** questions*

*There are **twelve** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathbb{C}$ be a small category.

(i) State and prove the Yoneda Lemma.

(ii) Define the Yoneda embedding $H_\bullet$ and show that it is full and faithful.

(iii) Show that $H_\bullet(f)$ is monic if and only if f is monic.

(iv) Show that $H_\bullet(f)$ is epic if and only if f is split epic. [*Recall: a split epic is a morphism e such that $eg = 1$ for some morphism g.*]

2 Suppose that

$$\begin{tikzcd} \cdot \arrow[r, "b"] \arrow[d, "a"'] & \cdot \arrow[r, "c"] \arrow[d, "g"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "f"'] & \cdot \arrow[r, "e"'] & \cdot \end{tikzcd}$$

is a commutative diagram.

(i) Show that if both small squares are pullbacks then so is the large rectangle.

(ii) Show that if the large rectangle and the right hand square are pullbacks, then so is the left hand square.

(iii) State carefully a result to the effect that the pullback of a pullback square is a pullback square, and prove it.

3 **(i)** Show that if a category has finite products and equalizers then it has all finite limits.

(ii) Show that a category with a terminal object and pullbacks has binary products and equalizers; deduce that such a category has all finite limits.

4 **(i)** Give a definition of limits in terms of representability.

(ii) Suppose that $F : \mathbb{I} \times \mathbb{J} \longrightarrow \mathcal{D}$ is such that the functors $F_J = (-, J) : \mathbb{I} \longrightarrow \mathcal{D}$ have limits in D for all $J \in \mathbb{J}$. Show that the assignment

$$J \longmapsto \int_I F(I, J)$$

extends to a functor

$$\int_I F(I, -) : \mathbb{J} \longrightarrow \mathcal{D}$$

and explain in what sense this functor is unique. (Standard facts about representability may be assumed.)

(iii) Suppose in addition that the functor $\int_I F(I, -)$ has a limit. Show that $F : \mathbb{I} \times \mathbb{J} \longrightarrow \mathcal{D}$ has a limit.

(iv) State precisely a Fubini Theorem to the effect that limits commute with limits, and prove it.

5 **(i)** Explain the notions of ends and coends.

(ii) Prove the Density Formula

$$X(U) \cong \int^W \mathbb{C}(U, W) \times X(W)$$

for a presheaf $X \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$.

(iii) Deduce that every presheaf is a colimit of representables.

6 Let $\mathcal{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{D}$ be an adjunction.

(i) Define the *unit* η and the *counit* ε of the adjunction.

(ii) Prove the triangle identities for η and ε.

(iii) Prove that given functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and natural transformations $1 \xrightarrow{\eta} GF$, $FG \xrightarrow{\varepsilon} 1$ satisfying the triangle identities, there is a unique adjunction between F and G with η as its unit and ε as its counit.

7 Explain how a poset is realized as a category. What is a functor between posets?

(i) Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ be a pair of functors between posets. Describe in explicit terms what an adjunction $f \dashv g$ is.

(ii) Let $p : A \longrightarrow B$ be a map of sets, and let $p^* : \mathcal{P}(\mathcal{B}) \longrightarrow \mathcal{P}(\mathcal{A})$ be the induced map of power sets. Exhibit left and right adjoints to p^*.

(iii) Fix a nonempty topological space S, and let $\mathcal{O}(S)$ be the poset of open subsets of S, ordered by inclusion. Let

$$\Delta : \mathbf{Set} \longrightarrow [\mathcal{O}(S)^{\mathrm{op}}, \mathbf{Set}]$$

be the functor assigning to a set A the presheaf ΔA with constant value A. Exhibit left and right adjoints to Δ. (*In this part you are only expected to define the functors on objects, and when you show adjointness you are not expected to carry out any formal checks of naturality.*)

8 Suppose the functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and the natural transformations $1 \xrightarrow{\eta} GF$, $FG \xrightarrow{\varepsilon} 1$ are given such that the composite

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

is the identity.

Show that the composite

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

is an idempotent (in $[\mathcal{C}, \mathcal{D}]$). Show that G has a left adjoint if and only if this idempotent splits.

[*Recall: An idempotent is a map $e : X \longrightarrow X$ such that $e^2 = e$. An idempotent $e : X \longrightarrow X$ splits just when there are maps $i : Y \longrightarrow X$ and $r : X \longrightarrow Y$ such that $ri = 1_Y$ and $ir = e$.*]

9 **Either** state and prove the General Adjoint Functor Theorem. (If you wish to appeal to an initial object lemma you should prove it.)

Or state and prove the Special Adjoint Functor Theorem. (You may assume the General Adjoint Functor Theorem.)

10 (i) Define the structure of a monad on a category $\mathcal{C}$, and the category of algebras for a monad.

(ii) Show that the forgetful functor from the category $\mathcal{C}^T$ of T-algebras to $\mathcal{C}$ has a left adjoint.

(iii) Show that any T-algebra is a coequalizer of a diagram of free algebras.

11 (i) Let $G : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor. What is a G-split-coequalizer pair? What does it mean for G to reflect G-split-coequalizers?

(ii) Suppose that T is a monad on $\mathcal{C}$ induced by an adjunction $F \dashv G : \mathcal{D} \longrightarrow \mathcal{C}$. Define the comparison functor $K : \mathcal{D} \longrightarrow \mathcal{C}^T$. Show that K is full and faithful if G reflects G-split-coequalizers.

12 (i) Define the notion of a *monoidal category*. In what sense is a monoidal category a bicategory?

(ii) Show that **Set** has the structure of a monoidal category. Is this structure the only monoidal structure on **Set**? Justify your answer.