UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 9 to 12

PAPER 2

FURTHER CHARACTER THEORY

Attempt ***THREE*** *questions*

There are **five** *questions in total*

The questions carry equal weight

In questions of two parts the first(a) makes up for 30% and the second(b) for 70% of the total credit.

You may use Clifford's theorem in a proof without stating it explicitly; however, you should make clear what you mean by any other result from the course.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Describe Brauer's characterisation of characters. (Define all concepts needed which are not obvious.)

(b) Let G be a finite group and A, B subsets of G such that $G = A \cup B \cup \{1\}$ is a disjoint union. Suppose that if $a \in A$ and $b \in B$ then $(o(a), o(b)) = 1$. Show that there exists a generalised character ϑ of G with $\vartheta(a) = 1$ if $a \in A$ and $\vartheta(b) = 0$ if $b \in B$.

2 (a) Define the following terms: i) Frobenius group, ii) Frobenius kernel, iii)Frobenius complement. State Brauer's Permutation Lemma.

(b) Prove that the following two statements are equivalent for a normal subgroup $1 < N < G$.

(i) N is the Frobenius kernel of the Frobenius group G.

(ii) For all $\varphi \in \mathrm{Irr}(N) \setminus \{1_N\}$ we have $\varphi^G \in \mathrm{Irr}(G)$.

3 (a) Define what is meant by an M-group. State Taketa's theorem and Dade's theorem on embeddings.

(b) For a finite group G define $\mathcal{H} = \{H \leq G \mid \varphi^G \text{ is irreducible for some } \varphi \in \mathrm{Irr}(H)\}$. Show that if G is an M-group then the intersection of the elements of $\mathcal{H}$ is Abelian.

4 State Thompson's Theorem on extendibility. State and prove Isaacs' Lemma about p-nilpotency. With its help prove Isaacs' Theorem concerning the group algebras of nilpotent groups and Thompson's Theorem on the existence of normal p-complement.

5 (a) Let H be a subgroup of the finite group G. Let Ψ be a representation of H with corresponding character ψ. Describe the induced representation Ψ^G and the induced character ψ^G. Define the induced class function. Define $\det\psi$, the determinant-character of a character ψ of a finite group G.

(b) State and prove Frobenius' Reciprocity Theorem. Show that if $P \in \mathrm{Syl}_p(G)$ is Abelian then p does not divide $|G' \cap Z(G)|$. (*Hint:* By contradiction take $Q \leq P \cap G' \cap Z(G)$ of order p and take $\lambda \in \mathrm{Irr}(Q) \setminus \{1_Q\}$).