UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 7 June 2002 9 to 12

# PAPER 19

# KNOT THEORY

*Attempt* ***THREE*** *questions*

*There are* ***five*** *questions in total*

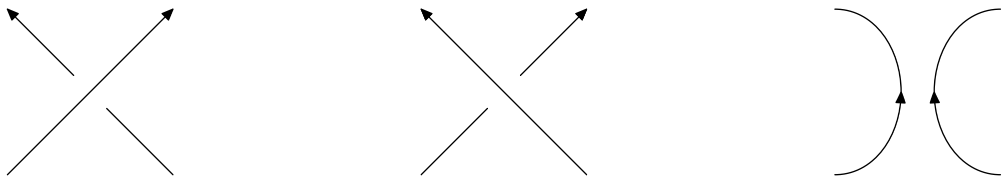
*The questions carry equal weight*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Define the Kauffman Bracket (Laurent) polynomial $\langle D \rangle$ of a diagram $D$ of a classical link. Investigate the the way that $\langle D \rangle$ changes when $D$ is changed by a Reidemeister move and define the Jones polynomial $V(L)$ of an oriented link $L$.

Prove that whenever three oriented links $L_+$, $L_-$ and $L_0$ are the same, except in the neighbourhood of a point where they are as shown in the following diagram, then

$$t^{-1}\, V(L_+) - t\, V(L_-) + (t^{-1/2} - t^{1/2})\, V(L_0) = 0\ .$$



$L_+$ $L_-$ $L_0$

Are the following statements about the Jones polynomial true or false? For each give a proof or a counter example.

(a) If $K_1$ and $K_2$ are oriented knots then $V(K_1)V(K_2) = V(K_1 + K_2)$.

(b) Any element of $\mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$ is the Jones polynomial of *some* oriented link $L$.

(c) If $L$ has an $n$-crossing diagram then the breadth of $V(L)$ is at most $n$.

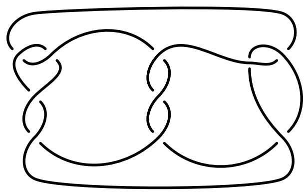
(d) For any oriented knot $K$, the evaluation of $V(K)$ at $t = -1$ is equal to $\Delta_K(-1)$ where $\Delta_K(t)$ is the Conway-normalised Alexander polynomial of $K$. [General results about $\Delta_K(t)$ may be quoted without proof.]

**2** Explain what is meant by the *sum* of two oriented knots and what is meant by saying that a knot is *prime*. Define the genus $g(K)$ of a knot $K$, proving that such a genus does indeed exist, and prove that, for any two oriented knots $K_1$ and $K_2$, the genus satisfies $g(K_1) + g(K_2) = g(K_1 + K_2)$. Deduce that any knot can be expressed as a sum of prime knots.

Explain briefly why this deduction cannot be made using, instead of the idea of the genus, (i) the idea of crossing number, or (ii) the breadth of the Jones polynomial, or (iii) the breadth of the Alexander polynomial.

Prove that the expression of a knot as the sum of prime knots is unique up to ordering of the summands.

**3** What is a Seifert matrix for a knot $K$ in $S^3$? By considering the infinite cyclic cover of the complement of $K$, define the Alexander module and the Alexander polynomial $\Delta_K(t)$ of $K$. Explain, with detailed proof, how $\Delta_K(t)$ can be calculated from any Seifert matrix for $K$. [General results about covering spaces, and results about finitely generated modules, may be assumed without proof.]



If now $K$ is the knot $10_{145}$ shown above, prove that, up to multiplication by a unit,

$$\Delta_K(t) = t^4 + t^3 - 3t^2 + t + 1.$$

What is the genus of $K$? Is $K$ a prime knot? By quoting the theory of the signature of a knot, or otherwise, determine whether $K$ is equivalent to its reflection $\overline{K}$.

**4** Explain without proof how to write down a presentation of the group of a knot, namely the fundamental group of the complement of the knot, based on a planar diagram of the knot. Show that the group of a trefoil knot $T$ (the knot $3_1$ ) has a presentation with two generators and one relator.

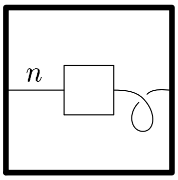
Suppose that a knot group $G$ has a presentation with generators $\{x_1, x_2, \ldots, x_n\}$ and relators $\{r_1, r_2, \ldots, r_m\}$. Describe in detail the Fox free differential calculus that associates, to the group presentation, a matrix $\left(\frac{\partial r_i}{\partial x_j}\right)$ and explain, giving full proofs, how this matrix is related to the Alexander module of the knot.

Determine the Alexander polynomial of $T$. Prove that the knot $T + T$ cannot have a presentation of its knot group consisting of just two generators and one relator.

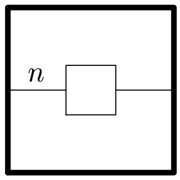
[General results about covering spaces, and results about finitely generated modules, may be assumed without proof.]

**5** Explain the idea of *skein theory* associated with oriented 3-manifolds with boundary. Define the Temperley-Lieb algebras, with parameter a fixed complex number $A$, and develop the theory of the Jones-Wenzl idempotents in these algebras.

Prove the identity in the $n$th Temperley-Lieb algebra depicted below where, using the usual conventions, the small square represents the Jones-Wenzl idempotent.




$$= (-1)^n A^{n^2+2n}$$




Explain, with proofs, how this skein theory can be used to give, for suitable choices of $A$, well defined 'quantum' invariants of closed oriented 3-manifolds associated to the theory of the Jones polynomial.

[General results about 3-manifolds may be quoted.]