UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 31 May 2002 1.30 to 3.30

PAPER 18

COMPACT LIE GROUPS

Attempt ***THREE*** *questions*

There are ***four*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove the *Weyl Integration Formula* for a compact, connected Lie group G with maximal torus T, explaining the meaning of all terms that appear:

$$\int_G \varphi(g)dg = \frac{1}{|W|}\int_T \varphi(t)\cdot|\Delta(t)|^2\,dt$$

[*You may use without proof the main results about maximal tori and conjugacy classes in compact Lie groups, provided you state them clearly*]

2 Prove the irreducibility of all symmetric powers of the standard representation $\mathbb{C}^n$ of $U(n)$. You may use the Weyl *character* formula without proof, but if you use the Weyl *dimension* formula, you must prove it first.

3 State and prove the *Weyl character formula* for $U(n)$. You may assume the Weyl Integration Formula and the Schur orthogonality relations.

4 A sub-torus T of $SO(5)$, parametrized by $x_1, x_2 \in [0, 2\pi)$, consists of the following matrices:

$$\begin{bmatrix} \cos x_1 & \sin x_1 & 0 & 0 & 0 \\ -\sin x_1 & \cos x_1 & 0 & 0 & 0 \\ 0 & 0 & \cos x_2 & \sin x_2 & 0 \\ 0 & 0 & -\sin x_2 & \cos x_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\mathbb{C}^5 = \mathbb{R}^5 \otimes \mathbb{C}$ be the standard representation of $SO(5)$.

(a) Using x_1 and x_2 as coordinates in the Lie algebra $\mathfrak{t}$ of T, show that the T-weights of the representation $\Lambda^2\mathbb{C}^5$ are: $\pm i(x_1 + x_2)$, $\pm i(x_1 - x_2)$, $\pm ix_1$ and $\pm ix_2$, each occurring with multiplicity 1, and additionally, 0, occurring with multiplicity 2.

(b) Show that $T \subset SO(5)$ is a maximal torus, and determine the roots of the Lie algebra $\mathfrak{so}(5)$.

[*Hint: Prove that the adjoint representation of* $SO(5)$ *is isomorphic to* $\Lambda^2\mathbb{R}^5$]

(c) Describe the Weyl group, assuming that its action on $\mathfrak{t}$ is generated by reflections about the root hyperplanes. Find representatives in $SO(5)$ of some typical Weyl group elements.

(d) Prove that $\Lambda^2\mathbb{C}^5$ is an irreducible representation of $SO(5)$.

[*You may wish to find its highest weight and use the Weyl dimension formula, after you identify all terms that appear in it:*

$$\dim L_\lambda = \prod_{\alpha>0} \frac{\langle \lambda + \rho | \alpha \rangle}{\langle \rho | \alpha \rangle}]$$