UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Monday 3 June 2002   1.30 to 4.30

PAPER 17

ALGEBRAIC TOPOLOGY

*Attempt **THREE** questions, one of which should be question 4 **or** question 5*

*There are **five** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Give a careful definition of a *finite* CW-complex and show that it is Hausdorff. Show further that the weak topology coincides with the topology induced by inclusion in some Euclidean space.

By exhibiting a suitable cellular decomposition of the real projective space $\mathbb{R}P^n$ calculate the homology groups with respect to the coefficients $\mathbb{R}, \mathbb{Z}$ and $\mathbb{Z}/2$.

2 Define the term 'Serre Fibration'. If PY denotes the space of paths in Y with initial point y_0 and $p : PY \to Y$ sends a path in Y to its end point, show that $p : PY \to Y$ is a Serre fibration.

Write down the long sequence of homotopy groups for the CW-pair (E, F) and prove that it is exact at the point $\pi_n(E, F)$. If $p : E \to B$ is a Serre fibration with $F = p^{-1}(b_0)$ show that $p_* : \pi_n(E, F) \to \pi_a(B, b_0)$ is an isomorphism for all $n \geqslant 1$.

If U_n denotes the unitary group of complex $n \times n$ matrices, and U_n is topologised as a subset of $\mathbb{C}^{n^2}$ show that U_n/U_{n-1} is homeomorphic to S^{2n-1}. Show that for $i \leqslant 2n-1, \pi_i(U_n) \cong \pi_i(U_{n+1})$.

3 Let X be a CW-complex such that $\pi_i(X) = 0$ for $i < n-1$, and let ΣX be its suspension. Show that the natural homomorphism $\pi_{i-1}(X) \to \pi_i(\Sigma X)$ is surjective for $i \leqslant 2n-2$ and bijective for $i < 2n-2$.

Quoting any additional result which you need, show that

$$\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z} \text{ for all } n \geqslant 1.$$

4 Let A be a not necessarily commutative ring with unit. Define $K_0(A)$ and $K_1(A)$ and give conditions under which the array of rings

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"'] & A_1 \arrow[d, "j_1"] \\ A_2 \arrow[r, "j_2"'] & A' \end{tikzcd}$$

is a Milnor square. Prove the exactness of the sequence

$$K_1A \to K_1A_1 \oplus K_1A_2 \to K_1A' \to K_0A \to K_0A_1 \oplus K_0A_2 \to K_0A'.$$

Let $\mathbb{Z}[C_p]$ denote the group ring of the cyclic group C_p (p=prime) and $\mathbb{Z}[\zeta]$ the integral domain obtained by adjoining $\zeta = e^{2\pi i/p}$ to $\mathbb{Z}$. Using a suitable Milnor square show that

$$K_0\mathbb{Z}[C_p] \cong K_0\mathbb{Z}[\zeta].$$

Hint: Consider the units $u = \frac{\zeta^k - 1}{\zeta - 1}$ for $1 \leqslant k \leqslant p-1$.

5 Write an essay on half-exactness and the duality between 'homotopy' and 'cohomology' functors.