UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2002 1.30 to 4.30

# PAPER 16

# SYMPLECTIC GEOMETRY AND HAMILTONIAN SYSTEMS

*Attempt **FOUR** questions*

*There are **five** questions in total*

*The questions carry equal weight*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1 (a)** Define the group of Hamiltonian diffeomorphisms.

**(b)** Let $(M, \omega)$ be a closed symplectic manifold such that $H^1(M, \mathbf{R}) = 0$. Show that the identity component of the group of symplectomorphisms coincides with the group of Hamiltonian diffeomorphisms.

**2 (a)** Define Hofer's metric $\rho_\infty$ and the displacement energy.

**(b)** Prove that the displacement energy of $S^{2n-2} \subset \mathbf{R}^{2n-1} \subset \mathbf{R}^{2n}$ with respect to the Hofer metric $\rho_\infty$ vanishes.

**3 (a)** Define the Liouville class of a Lagrangian submanifold $L$ in $\mathbf{R}^{2n}$ endowed with the canonical symplectic form. State the condition under which $L$ is rational and define the invariant $\gamma(L)$ associated with $L$.

**(b)** Let $L$ be a closed rational Lagrangian submanifold contained in $\mathbf{R}^{2n}$. Using that $e(L) \geqslant \gamma(L)/2$ show that Hofer's metric in $\mathbf{R}^{2n}$ is nondegenerate.

**4 (a)** Show that the graph of a 1-form $\alpha$ on $N$ is a Lagrangian submanifold of $T^*N$ if and only $\alpha$ is closed.

**(b)** Consider the map $f : T^*N \to T^*N$ given by $f(q,p) = (q, p + \alpha(q))$ where $q \in N$ and $p \in T^*_q N$. Show that if $\alpha$ is closed then $f$ is a symplectomorphism.

**(c)** Show that if $\alpha$ is exact then $f$ is a Hamiltonian diffeomorphism.

(Hint: $f$ is a Hamiltonian diffeomorphism if and only if $f^*\theta - \theta$ is exact, where $\theta$ is the canonical 1-form of $T^*N$).

**5** Let $L$ be a closed Lagrangian submanifold of $\mathbf{C}^n$ endowed with the canonical symplectic form. Suppose that there exists a smooth map $h : (D^2, \partial D^2) \to (\mathbf{C}^n, L)$ such that $\overline{\partial} h = 0$ (i.e. $h$ defines a holomorphic disc with Lagrangian boundary conditions). Show that if $h$ is not constant then the Liouville class of $L$ is nonzero.