UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 9 to 12

PAPER 15

GEOMETRIC INVARIANT THEORY

*Attempt **FOUR** questions*

*There are **five** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Compute the ring of invariants of the cyclic group of order n with generator σ acting on the affine plane A^2 by

$$\sigma(x,y) = (\zeta x, \zeta^{-1} y),$$

where ζ is a primitive n^{th} root of unity in the complex numbers, for a prime number $n \geq 3$.

Show that the quotient variety is smooth outside the image of the origin.

2 (a) Show that any homomorphism of complex algebraic groups from the multiplicative group G_m to itself has the form $x \mapsto x^i$ for some integer i.

(b) Give any example of a sub-$\mathbb{C}$-algebra of a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ which is not finitely generated.

3 Let $X = P((M_n\mathbb{C})^*)$ be the projective space of lines in the vector space $M_n\mathbb{C}$ of $n \times n$ matrices. Consider the action of $SL(n,\mathbb{C})$ on X by conjugation of matrices. Use the Hilbert-Mumford theorem to find the set of $SL(n,\mathbb{C})$-semistable points in X.

4 (a) If E is a holomorphic vector bundle on a smooth compact complex curve, show that the slopes of all subbundles of E are bounded above.

(b) Show by an example that the slopes of the subbundles of E need not be bounded below.

5 Let V be a unitary representation of the fundamental group $\pi_1 X$ of a smooth compact complex curve X. Let E_V be the corresponding holomorphic vector bundle on X.

(a) Show that $H^0(X, E_V) = V^{\pi_1, X}$, the subspace of V fixed by π_1, X.

(b) Show that E_V is semistable of degree 0.