UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 9 to 12

PAPER 12

TORIC VARIETIES

Attempt ***THREE*** *questions*

There are ***four*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $N \cong \mathbf{Z}^n$ denote a lattice and $\sigma \subset N_{\mathbf{R}} = N \otimes \mathbf{R}$ a rational polyhedral cone with a vertex. Show that there is an order reversing bijection between the faces of σ and those of the dual cone $\check{\sigma} \subset M_{\mathbf{R}} = M \otimes \mathbf{R}$, where M denotes the dual lattice to N.

Given a face $\tau = \sigma \cap \mu^{\perp}$ of σ ($\mu \in M \cap \check{\sigma}$), show that $X_{\check{\tau}}$ is an open subvariety of $X_{\check{\sigma}}$. Prove that $X_{\check{\sigma}} \backslash X_{\check{\tau}}$ is the union of closed subvarieties X_{γ}, for γ ranging over faces of $\check{\sigma}$ not containing μ. For faces τ_1, τ_2 of σ, show that

$$X_{(\tau_1 \cap \tau_2)\check{}} = X_{\check{\tau}_1} \cap X_{\check{\tau}_2}.$$

Suppose now that Σ is a fan in $N_{\mathbf{R}}$, with corresponding toric variety X_{Σ}. For $\tau \in \Sigma$, define the closed subvariety F_{τ} of X_{Σ} as the closure of the subset $X_{\tau^{\perp}}$. Prove that

$$X_{\Sigma} \backslash X_{\check{\sigma}} = \bigcup_{\substack{\tau \in \Sigma \\ \tau \not\subset \sigma}} F_{\tau}.$$

2 State necessary and sufficient conditions on a fan Σ in $N_{\mathbf{R}}$, where $N \cong \mathbf{Z}^n$, for the toric variety $X_{\Sigma,N}$ to be smooth. Show that, for any given fan Σ, there is a subdivision Σ' with $X_{\Sigma',N}$ smooth.

Let

$$\sigma = \mathbf{R}_{+}(1,0,0) \ + \ \mathbf{R}_{+}(1,1,1) \ + \ \mathbf{R}_{+}(0,2,1) \ + \ \mathbf{R}_{+}(1,1,0)$$

in $N_{\mathbf{R}}$, where now $N = \mathbf{Z}^3$. Show that $X_{\check{\sigma}}$ has an isolated singularity at the origin. Find two *different* smooth resolutions $X_{\Sigma_1} \to X_{\check{\sigma}}$ and $X_{\Sigma_2} \to X_{\check{\sigma}}$ of the singularity $X_{\check{\sigma}}$, both of which are minimal (in the sense that neither Σ_i is the subdivision of a fan Σ' with $X_{\Sigma',N}$ smooth).

Describe the exceptional surfaces (if any) for both the smooth resolutions $X_{\Sigma_i} \to X_{\check{\sigma}}$ $(i = 1, 2)$. Show that the birational map

$$X_{\Sigma_1} - \to X_{\Sigma_2}$$

may be factored into a composite of toric blow-ups and inverses of toric blow-ups.

3 Prove that any complex affine toric surface singularity has the form

$$X_{r,q} = \mathbf{C}^2/\mu_r \, ,$$

where r, q are coprime integers such that $0 < q < r$ and $\mu_r = \langle \zeta \rangle$ acts by

$$(x, y) \mapsto (\zeta^q x, \zeta y),$$

with ζ denoting a primitive rth root of unity.

Describe (with proof) the recipe for obtaining a (minimal) resolution of $X_{r,q}$, and explain how to find the number t of exceptional curves in the resolution. Prove that $t = 1$ if and only if $q = 1$, and that $t = 2$ if and only if $q > 1$ and q divides $r + 1$.

Classify all complete complex toric surfaces X_Σ with $|\Sigma^{(1)}| = 3$ containing at most one singularity. Show that there is a complete complex toric surface X_Σ with $|\Sigma^{(1)}| = 4$ containing $X_{r,q}$ as its only singular point if and only if $r - 1$ has a factor s congruent to $\pm q$ mod r.

4 For $X_{\Sigma,N}$ a complete toric variety, with $\Sigma^{(1)} = \{\tau_1, \ldots, \tau_d\}$, let F_i denote the invariant divisors F_{τ_i}, for $i = 1, \ldots, d$. Describe the maps in the standard short exact sequence

$$0 \to M \to \bigoplus_{i=1}^{d} \mathbf{Z} F_i \to \mathrm{Cl}(X_\Sigma) \to 0$$

(you need not prove exactness). If $D = \sum r_i F_i$ is a divisor on $X_{\Sigma,N}$, explain (without proof) how to find a basis of the vector space $\mathcal{L}(D)$, and state (without proof) a numerical criterion for D to be ample.

Give a toric definition for the weighted projective space $X = \mathbf{P}(q_0, \ldots, q_n)$ (specify a lattice N and a fan Σ). Assuming that X is well-formed, show that $\mathrm{Cl}(X)$ is isomorphic to $\mathbf{Z}$, and exhibit an explicit ample generator. If D is an ample divisor on X with class generating $\mathrm{Cl}(X)$, show that

$$\bigoplus_{r \geqslant 0} \mathcal{L}(rD) \cong S(Q)$$

as graded k-algebras, where $S(Q)$ denotes the polynomial ring $k[X_0, \ldots, X_n]$ over the field of definition k, graded by specifying that $\deg(X_i) = q_i$ for $0 \leqslant i \leqslant n$.