

MATHEMATICAL TRIPOS      Part III

---

Wednesday 5 June 2002   9 to 11

---

PAPER 11

ALGEBRAIC METHODS IN COMBINATORICS

*Attempt **TWO** questions*

*There are **three** questions in total*

*The questions carry equal weight*

You may not start to read the questions  
printed on the subsequent pages until  
instructed to do so by the Invigilator.

**1 (i)** Prove that, for any positive integer  $k$ , there exists a  $(v, k, 1)$ -design for infinitely many values of  $v$ . (You may assume that the multiplicative group of a finite field is cyclic.)

**(ii)** Show that if the edge-set of the complete graph  $K_n$  can be partitioned into edge-disjoint copies of the complete bipartite graph  $K_{k,k}$  then  $k^2$  divides  $n - 1$ .

**2** Write an essay on the connections between the eigenvalues and the expansion property of a graph, the topic including also an explicit construction of a family of expanders.

**3** State the Combinatorial Nullstellensatz. Deduce the following results.

**(i)** For any integer  $k \geq 3$  there is a constant  $c_k$  such that any graph on  $n$  vertices without a  $k$ -regular subgraph has at most  $c_k n \log n$  edges.

**(ii)** Let  $p$  be a prime and  $A, B$  be two non-empty subsets of the finite field  $\mathbb{F}_p$  with  $|A| > |B|$ . Show that the set

$$C = \{x + y \mid x \in A, y \in B, x \neq y\}$$

has at least  $\min\{p, |A| + |B| - 2\}$  elements.

[Hint: consider the polynomial  $f(x, y) = (x - y) \prod_{z \in C} (x + y - z)$ .]