UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 1.30 to 3.30

PAPER 10

RAMSEY THEORY

Attempt ***THREE*** *questions*

There are ***four*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let c be a colouring of $\mathbb{N}$ using (possibly) infinitely many colours, and let $m \in \mathbb{N}$. Given $x_1, \ldots, x_m \in \mathbb{N}$, let $e(x_1, \ldots, x_m)$ denote the equivalence relation $\sim$ on $\{1, \ldots, m\}$ given by

$$i \sim j \iff c(x_i) = c(x_j), \quad 1 \leqslant i, j \leqslant m,$$

and define a (finite) colouring c' of $\mathbb{N}^2$ by setting

$$c'(a, d) = e(a, a + d, a + 2d, \ldots, a + (m-1)d), \quad a, d \in \mathbb{N}^2.$$

By applying Gallai's theorem, deduce that there is an arithmetic progression of length m on which c is either constant or injective.

2 State and prove van der Waerden's theorem. Deduce that, if $a_1, \ldots, a_n$ are non-zero rationals, then the matrix $(a_1, \ldots, a_n)$ is partition regular if and only if some (non-empty) subset of the a_i has sum 0.

[No form of Rado's theorem may be assumed without proof.]

3 What is an *ultrafilter* on $\mathbb{N}$? Prove that there exists a non-principal ultrafilter on $\mathbb{N}$. Define the topological space $\beta\mathbb{N}$, and prove that it is compact and Hausdorff.

Explain carefully how the operation $+$ on $\beta\mathbb{N}$ is defined, and prove that $+$ is associative and left-continuous.

State Hindman's theorem, and show how to deduce it from the existence of an idempotent for $+$ on $\beta\mathbb{N}$. (You are not required to prove that an idempotent exists.)

[You may assume simple properties of ultrafilters and their quantifiers].

4 What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? Prove that every $*$-Borel subset of $\mathbb{N}^{(\omega)}$ is Ramsey.