UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2001 9 to 12

PAPER 9

THE VALUE DISTRIBUTION OF ANALYTIC FUNCTIONS

Answer any ***THREE*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the Poisson–Jensen formula.

Let (z_n) be an infinite sequence of distinct points in the **simply-connected** domain $\Omega \subset \mathbb{C}$. Show that the following conditions are equivalent:

(i) There is a bounded, analytic function $f : \Omega \to \mathbb{C}$ with zeros at each of the points (z_n) and nowhere else in Ω.

(ii) The domain Ω has a hyperbolic metric ρ and, for any $w \in \Omega$, the series

$$\sum \exp\{-\rho(w, z_n)\}$$

converges.

By considering the punctured disc $\{z \in \mathbb{C} : 0 < |z| < 1\}$, or otherwise, show that these conditions need not be equivalent when Ω is not simply-connected.

2 State Nevanlinna's First and Second Fundamental Theorems for a function meromorphic on the unit disc.

Prove that a meromorphic function $f : \mathbb{D} \to \mathbb{C}_\infty$ has bounded characteristic if and only if f is the ratio of two bounded analytic functions on the unit disc.

Let $g : \mathbb{D} \to \mathbb{C}$ be an analytic function. Show that the characteristic of g may be bounded without g itself being bounded.

Suppose that, for some p with $1 < p < \infty$ and some constant C, we have

$$\int_0^{2\pi} (1 + |g(re^{i\theta})|^2)^{p/2} \frac{d\theta}{2\pi} < C \qquad \text{for} \qquad 0 < r < 1 \ .$$

By using the fact that the logarithm is concave, or otherwise, show that g must have bounded characteristic.

3 Let $A = \{z \in \mathbb{C} : 0 < |z| < 2\}$ and let $f : A \to \mathbb{C}_\infty$ be a meromorphic function. For each $a \in \mathbb{C}_\infty$ that does not lie on the curve $\{f(z) : |z| = 1\}$, define

$$N(r; a) = \sum \left\{ \left(\log \frac{|z|}{r} \right) \deg f(z) \ : \ f(z) = a \text{ and } 1 > |z| \geqslant r \right\};$$

$$m(r; a) = \int_0^{2\pi} \log \frac{k(f(e^{i\theta}), a)}{k(f(re^{i\theta}), a)} \, d\theta;$$

$$T(r) = \frac{1}{4\pi} \int_{\{1>|z|>r\}} \left(\log \frac{|z|}{r} \right) \left(\frac{2|f'(z)|}{1 + |f(z)|^2} \right)^2 dx \, dy;$$

where $z = x + iy$ and $0 < r \leqslant 1$. Show that there is a function $I(a)$, independent of r, with

$$T(r) = I(a) \log \frac{1}{r} + N(r; a) + m(r; a) \ .$$

Prove that

$$\liminf_{r \to 0} \frac{T(r)}{\log 1/r}$$

is finite if and only if f has a removable singularity (or a pole) at 0.

4 State Ahlfors' Second Fundamental Theorem.

Show how to derive from this Ahlfors' Five Islands Theorem for meromorphic functions $f : \mathbb{C} \to \mathbb{C}_\infty$. Give an example to show that there is no corresponding Four Islands Theorem.

5 Let S be the unit sphere in $\mathbb{R}^3$. Let $L(\gamma)$ be the length of a curve γ on S and $\mathbb{A}(U)$ the area of $U \subset S$, both relative to the spherical metric. Prove that there is a constant κ with

$$\mathbb{A}(U)\mathbb{A}(U') \leqslant \kappa L(\partial U)^2$$

for all subsets U of S with complement $U' = S \setminus U$ and a smooth boundary ∂U.

Let $i : V \hookrightarrow S$ be the inclusion map for a compact subset V of S which has a smooth boundary. Show that, for any non-empty, open subset D of S we have

$$\left| \frac{\mathbb{A}(V)}{\mathbb{A}(S)} - \frac{\mathbb{A}(V \cap D)}{\mathbb{A}(D)} \right| \leqslant C \frac{L(\partial V)^2}{\mathbb{A}(S)\mathbb{A}(D)}$$

for some constant C.

Now consider another smooth Riemannian metric on S and let $\widetilde{\mathbb{A}}(U)$ denote the area of a set U relative to this new metric. Show that there is a smooth function $\theta : S \to (0, \infty)$ with

$$\widetilde{\mathbb{A}}(U) = \int_U \theta(x)\, d\mathbb{A}(x) = \int_0^\infty \mathbb{A}(U \cap D_t)\, dt$$

for $D_t = \{x \in S : \theta(x) > t\}$. Hence show that

$$\left| \frac{\mathbb{A}(V)}{\mathbb{A}(S)} - \frac{\widetilde{\mathbb{A}}(V)}{\widetilde{\mathbb{A}}(S)} \right| \leqslant C' \frac{L(\partial V)^2}{\mathbb{A}(S)\widetilde{\mathbb{A}}(S)}$$

for some constant C'.