UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 1.30 to 4.30

PAPER 77

SYMMETRIC FUNCTIONS

*Attempt any **FIVE** questions. The questions carry equal weight.*

The following notation is used throughout the paper. Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates, and let $n \in \mathbb{N}$. The set of all homogeneous symmetric functions of degree n over $\mathbb{Q}$ is denoted by $\wedge^n$. The vector space direct sum $\wedge = \wedge^0 \oplus \wedge^1 \oplus \ldots$ is the $\mathbb{Q}$-algebra of symmetric functions.

If λ is a partition of n, we write $|\lambda| = n$. The length of λ will be denoted by $l(\lambda)$.

If S is any finite set, $\mathfrak{S}_S$ will denote the symmetric group of all permutations of S.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe the RSK algorithm. Show that this algorithm is a bijection between $\mathbb{N}$-matrices $A = (a_{ij})_{i,j\geqslant 1}$ of finite support and ordered pairs (P, Q) of semistandard Young tableaux of the same shape. Show also that under this correspondence

j occurs in P exactly $\sum_i (a_{ij})$ times

and

i occurs in Q exactly $\sum_j (a_{ij})$ times.

[Standard facts about insertion paths can be quoted without proof, if clearly stated.]

Suppose that in the RSK algorithm $A \overset{RSK}{\longrightarrow} (P, Q)$, the matrix A is symmetric (so $P = Q$). Show by induction, or otherwise, that tr(A) is the number of columns of P of odd length. [Standard symmetry properties of the RSK algorithm may be assumed.]

2 State and prove the Jacobi-Trudi identity, defining carefully all the terms. If $\mu \subseteq \lambda$, deduce an expression for the number of standard Young tableaux of shape λ/μ. [If you use a result about determinants you should state it clearly.]

3 State and prove 'Burnside's Lemma'. Given a finite set S, and for any subgroup G of $\mathfrak{S}_S$, show that $Z_G = F_G$ where Z_G is the cycle indicator and F_G is the pattern inventory (both of which you should define).

The <u>rank</u> of a finite group G acting transitively on a set T is defined to be the number of orbits of G acting in the obvious way on $T \times T$, i.e. $w.(s,t) = (w.s, w.t)$. Thus G is doubly transitive if and only if rank $G = 2$. Let χ be the character of the action of G on T. Show that $\langle \chi, \chi \rangle = \text{rank}G$.

4 Define the partition set Par and explain how it can be endowed with a partial order $\subseteq$ to become Young's lattice. Define also the dominance order $\leqslant$ and the reverse lexicographic order $\overset{R}{\leqslant}$ on the set Par(n). Define the elementary and complete homogeneous symmetric functions. Sketch a proof of the 'fundamental theorem of symmetric functions'.

Write $\wedge = \mathbb{Q}[e_1, e_2, \ldots]$ as a $\mathbb{Q}$-algebra generated by the elementary symmetric functions. If $\{h_\lambda : \lambda \in \text{Par}\}$ are the complete homogeneous symmetric functions, prove the existence of an involutary automorphism $\omega : \wedge \to \wedge$ such that $\omega(e_\lambda) = h_\lambda$ for all partitions λ.

Let λ be a partition of n of length l. Define the <u>forgotten symmetric function</u> f_λ by $f_\lambda = \epsilon_\lambda \omega(m_\lambda)$ where $\epsilon_\lambda = (-1)^{n-l}$ and m_λ is the monomial symmetric function. Let $f_\lambda = \sum_\mu a_{\lambda\mu} m_\mu$. Show that $a_{\lambda\mu}$ is equal to the number of distinct permutations $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ of the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that

$$\{\alpha, +\alpha_2 + \ldots + \alpha_i : 1 \leqslant i \leqslant l\} \supseteq \{\mu_1 + \mu_2 + \ldots + \mu_j : 1 \leqslant j \leqslant l(\mu)\}.$$

5 Let λ and μ be partitions such that $\mu \subseteq \lambda$. Define the semistandard tableau of (skew) shape λ/μ and show how we may regard it as a Young diagram of shape λ/μ. If λ/μ is a skew shape define the <u>skew Schur function</u> $s_{\lambda/\mu}$ and the <u>Schur function</u> s_λ of shape λ. Show that for any skew shape λ/μ, $s_{\lambda/\mu}$ is indeed a symmetric function.

Let λ, μ be partitions with $|\lambda| = |\mu|$. Suppose that the (λ, μ)th Kostka number $K_{\lambda\mu}$ is non-zero. Prove that $\lambda \geqslant \mu$ (in the dominance order) and also that $K_{\lambda\lambda} = 1$. Deduce that the Schur functions s_λ form a $\mathbb{Q}$-basis for $\wedge$ ($\lambda \in \text{Par}$).

6 Let $\lambda \in \text{Par}$ and suppose $l(\lambda) \leqslant n$. Let $\delta = (n-1, n-2, \ldots, 0)$. Quoting any result that you need, prove the bialternant relation

$$a_{\lambda+\delta}/a_\delta = s_\lambda(x_1, \ldots, x_n).$$

Let p be a prime, and let A_p denote the matrix $[\zeta^{jk}]_{j,k=0}^{p-1}$, where $\zeta = e^{2\pi i/p}$. Show that every minor of A_p is non-zero (i.e. that every square submatrix B of A_p is nonsingular).

Show finally that

$$h_r(x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k^{n-1+r} \prod_{i \neq k} (x_k - x_i)^{-1}$$

[*Hint: Consider a particular value for λ in the bialternant.*]

7 State and prove the Cauchy identity (the RSK algorithm may be assumed). Use the identity to demonstrate that the Schur functions form an orthonormal basis for $\wedge$. <u>State</u> the dual Cauchy identity.

Let $F(t) = \sum_{j \geqslant 0} f_j t^j$ be a formal power series, where $f_0 = 1$. Expand the product $F(t_1)F(t_2)\ldots$ as a linear combination of Schur functions $s_\lambda(t_1, t_2, \ldots)$. The coefficient of $s_\lambda(t_1, t_2, \ldots)$ will be denoted by s_λ^F. Equivalently, if R is a commutative ring containing $f_1, f_2, \ldots$ and $\varphi : \wedge \to R$ is the homomorphism defined by $\varphi(h_j) = f_j$, then $s_\lambda^F = \varphi(s_\lambda)$. We extend the definition of s_λ^F by defining $u^F = \varphi(u)$ for any $u \in \wedge_R$.

(a) Show that if $F(t) = \prod_{i \geqslant 1} (1 - x_i t)^{-1}$, then $s_\lambda^F = s_\lambda(x)$.

(b) What if $F(t) = \Pi_{i \geqslant 1}(1 + x_i t)$?

8 What does it mean to say that a skew shape λ/μ is connected? Define a border strip, B and define the height, $\mathrm{ht}(B)$ of B.

Let p_λ be the power sum symmetric function. Let α be a weak composition of n. Define a border-strip tableau of shape λ/μ and type α. Show that

$$s_\mu p_\alpha = \sum_\lambda \chi^{\lambda/\mu}(\alpha) s_\lambda,$$

where $\chi^{\lambda/\mu}(\alpha) = \sum_T (-1)^{\mathrm{ht}(T)}$, summed over all border-strip tableaux of shape λ/μ and type α. Restricting to n variables where $n \geqslant \ell(\lambda)$, deduce that $\chi^\lambda(\alpha) = [x^{\lambda+\delta}]p_\alpha a_\delta$ where $\delta = (n-1, n-2, \ldots 0)$ and a_δ is the Vandermonde determinant. Deduce also the Murnaghan-Nakayama rule. Show also that s_δ is a polynomial in the odd power sums $p_1, p_3, \ldots$, where δ is the "staircase shape" $\delta = (m-1, m-2, \ldots, 1)$.

Finally, let $0 \leqslant s \leqslant n-1$ and $|\lambda| = n$. Show that if $w \in \mathfrak{S}_n$ is an n-cycle, then

$$\chi^\lambda(w) = \begin{cases} (-1)^s & \text{if } \lambda = (n-s, 1^s) \\ 0 & \text{otherwise} \end{cases} .$$