UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 13 June 2001 9 to 12

PAPER 74

ALGEBRAIC NUMBER THEORY

Attempt ***FIVE*** *questions, including at least* ***ONE*** *from each section.*

All questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 Give the definition of a Dedekind domain. Let $\mathfrak{o}$ be a Dedekind domain with field of fractions k. Let $\mathfrak{a}$ be a non-zero fractional ideal in k and define $\mathfrak{a}^{-1} = \{ x \in k \mid x\mathfrak{a} \subset \mathfrak{o} \}$. Show that $\mathfrak{a}^{-1}$ is a fractional ideal and that $\mathfrak{a}\mathfrak{a}^{-1} = \mathfrak{o}$.

2 (i) State and prove the Chinese remainder theorem for $\mathfrak{o}$ a Dedekind domain.

(ii) Let K/k be a normal extension of algebraic number fields. Let $\mathfrak{p}$ be a prime of k, whose factorisation in K is $\mathrm{conorm}_{K/k}\, \mathfrak{p} = \mathfrak{P}_1^{e_1} \ldots \mathfrak{P}_g^{e_g}$. Show that $\mathrm{Gal}(K/k)$ acts transitively on the $\mathfrak{P}_i$.

3 Let K/k be a finite extension of algebraic number fields. Define the relative ideal norm and prove that it is multiplicative. Let $\mathfrak{p}$ be a prime of k, whose factorisation in K is $\mathrm{conorm}_{K/k}\, \mathfrak{p} = \mathfrak{P}_1^{e_1} \ldots \mathfrak{P}_g^{e_g}$. Show that $[K : k] = \sum e_i f_i$ where f_i is the degree of $\mathfrak{O}/\mathfrak{P}_i$ over $\mathfrak{o}/\mathfrak{p}$.

[*Properties of the norm for elements may be assumed.*]

Section B

4 Let K/k be an extension of algebraic number fields. Let $\mathfrak{p}$ be a prime of k and $\mathfrak{P}$ a prime of K above $\mathfrak{p}$.

(i) Let $f(X)$ be a monic polynomial in $\mathfrak{o}_\mathfrak{p}[X]$ and suppose that the reduction mod $\mathfrak{p}$ factors as $\tilde{f}(X) = \phi_1(X)\phi_2(X)$ where ϕ_1, ϕ_2 in $(\mathfrak{o}/\mathfrak{p})[X]$ are coprime. Show that $f(X) = f_1(X)f_2(X)$ with $\tilde{f}_\nu(X) = \phi_\nu(X)$.

(ii) Suppose $\mathfrak{P}^e \mid\mid \mathfrak{p}$ and $\mathfrak{p} \mid e$. Show that if $\alpha \in \mathfrak{O}_\mathfrak{P}$ then $\mathrm{Tr}_{K_\mathfrak{P}/k_\mathfrak{p}}(\alpha) \in \mathfrak{p}_\mathfrak{p}$.

5 Let K/k be an extension of algebraic number fields. Define the relative different $\mathfrak{d}_{K/k}$. In the case $k = \mathbf{Q}$ describe the relationship with the discriminant d_K.

(i) For $K \supset L \supset k$ show that $\mathfrak{d}_{K/k} = \mathfrak{d}_{K/L}\mathfrak{d}_{L/k}$.

(ii) State a relationship between the different and ramification. Hence show that if K_1, K_2 are Galois over $\mathbf{Q}$ with coprime discriminants, then $[K_1K_2 : \mathbf{Q}] = [K_1 : \mathbf{Q}][K_2 : \mathbf{Q}]$.

Section C

6

Write an essay on the Hilbert class field. Illustrate by computing the Hilbert class field for *either* $\mathbf{Q}(\sqrt{-23})$ *or* $\mathbf{Q}(\sqrt{-30})$, explaining all necessary working.

[*The cubic* $X^3 + aX + b$ *has discriminant* $-4a^3 - 27b^2$.]

7 Let $m = m_1 m_2^2$ with m_1, m_2 coprime square-free positive integers. Suppose $m_1 \not\equiv \pm m_2 \pmod 9$. Show that $\mathbf{Q}(\sqrt[3]{m})$ has discriminant $-27m_1^2 m_2^2$. Find a unit in $\mathbf{Q}(\sqrt[3]{6})$ and show that this field has class number $h = 1$.

8 Let K/k be a quadratic extension of algebraic number fields with K totally complex and k totally real.

(i) Show that $[\mathfrak{O}_K^* : \mathfrak{o}_k^* \mu_K] = 1$ or 2, where $\mathfrak{O}_K^*$, $\mathfrak{o}_k^*$ are the unit groups in K, k, and μ_K is the group of roots of unity in K.

(ii) Show that the class number of k divides the class number of K.

[*You may assume any properties of the Hilbert class field you require.*]