UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2001 9 to 11

PAPER 73

STOCHASTIC NETWORKS

Attempt any ***THREE*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a closed migration process. Establish the form of the stationary distribution of a closed migration process.

A telephone switchboard has N incoming lines and one operator. Calls to the switchboard are initiated as a Poisson process of rate ν, but calls initiated when all N lines are in use are lost. A call finding a free line has then to wait for the operator to answer. The operator deals with waiting calls one at a time, and takes an exponentially distributed length of time with mean λ^{-1} to connect a call to the correct extension, after which the call lasts for an exponentially distributed length of time with mean μ^{-1}. All these lengths of time are independent of each other and of the initiating Poisson process. Model the system as a closed migration process, and show that in equilibrium the proportion of calls lost is

$$H(N)\left(\sum_{n=0}^{N} H(n)\right)^{-1},$$

where

$$H(n) = \left(\frac{\nu}{\lambda}\right)^{n} \sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^{i} \frac{1}{i!}.$$

2 Define the Erlang fixed-point approximation for a loss network with fixed routing, and establish the existence and uniqueness of the approximation.

Show, by means of an example or otherwise, that in a loss network with alternative routing the natural generalization of the Erlang fixed-point approximation may not be unique.

3 Outline a mathematical model of the slotted infinite-population ALOHA random access protocol, obtaining the recurrence

$$N_{t+1} = N_t + Y_{t-1} - I[Z_t = 1],$$

where $Z_t = 0, 1$ or $*$ according as $0, 1$ or more than 1 packets are transmitted in slot $(t, t+1)$, and Y_t is the number of arrivals in slot $(t, t+1)$. What does N_t represent?

Prove that for any positive arrival rate

$$P\{\exists J < \infty : Z_t = *, \text{ for all } t \geqslant J\} = 1.$$

Discuss whether we can expect a similar result for a finite-population model.

4 The dynamical system

$$\frac{d}{dt}x_r(t) = \kappa_r \left(w_r - x_r(t) \sum_{j \in r} \mu_j(t) \right) \qquad r \in R$$

$$\mu_j(t) = p_j \left(\sum_{s: j \in s} x_s(t) \right) \qquad j \in J$$

is proposed as a model for a communication network, where R is a set of routes, J is a set of resources, and $p_j(\cdot),\ j \in J$, are non-negative, continuous, increasing functions.

Provide a brief interpretation of this model, in terms of feedback signals generated by resources and acted upon by users.

By considering the function

$$U(x) = \sum_{r \in R} w_r \log x_r - \sum_{j \in J} \int_0^{\sum_{s: j \in s} x_s} p_j(y) dy$$

or otherwise, show that all trajectories of the dynamical system converge towards a unique point.