UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2001 1.30 to 3.30

PAPER 70

EARLY UNIVERSE COSMOLOGY

*Attempt any **TWO** questions. The questions are of equal weight.*

Candidates may make free use of a list of formulae and unit conversion factors which is included on a separate sheet.

When making cosmological estimates order of magnitude estimate will be accepted assuming all the correct physics has been accounted for.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a new inflation model based on the potential

$$V(\phi) = \alpha\left(\sigma^4 - 3\phi^4 + 2\phi^6/\sigma^2\right),$$

where $\alpha \ll 1$ is a small dimensionless number and $\sigma \ll m_{pl}$. Inflation can occur if the field rolls slowly from $\phi = 0$ towards its minimum at $\phi = \sigma$. Inflation ends when $\phi_{\rm end} \simeq \sigma$ and the field begins to oscillate around its minimum.

(a) The number of e-foldings in a time interval $[t_1, t_2]$ is given by

$$N = \int_{t_1}^{t_2} H dt\,.$$

Estimate the value of the field, ϕ_{50}, fifty e-folds before the end of inflation. Note that the precise value of $\phi_{\rm end}$ is not important, but that ${\phi_{50}}^2 \ll {\phi_{\rm end}}^2$.

(b) Relate the change in the wavenumber of modes k leaving the horizon during inflation to the motion of the inflaton down its potential; that is, relate $d\ln k$ to $d\phi$ assuming the Hubble constant H is approximately constant.

(c) Calculate the scalar and tensor spectral indices, n and n_T, fifty e-folds before the end of inflation, given that

$$n - 1 = \frac{d\ln \mathcal{P}_S}{d\ln k} \quad \text{and} \quad n_T = \frac{d\ln \mathcal{P}_T}{d\ln k},$$

where $\mathcal{P}_S$ and $\mathcal{P}_T$ are the power spectra of the scalar and tensor perturbations respectively.

(d) The ratio of tensor to scalar fluctuations, as reflected in the low multipole moments of the CMB, is given by $T/S \simeq -7n_T$. Find the tensor contribution to the large scale microwave background and comment on the prospects for observing it. Use the COBE DMR normalization to estimate the magnitude of α and discuss its size.

2

The Robertson-Walker metric for an open universe is given by

$$ds^2 = a^2(\tau)\left[d\tau^2 - \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right)\right],$$

where k is negative and is related to the curvature radius by $|k| \equiv r_{\rm curv}^{-2}$.

(a) Use Freidmann's equation to calculate $r_{\rm curv}$ in terms of Ω_m, the fraction of the critical density in matter, and the present Hubble constant, H_0.

(b) Use the metric to relate the angular scale subtended by an object at a comoving distance r_1 and comoving time τ_1 to its true physical size, D. You may assume that $\theta \ll 1$.

(c) Estimate the physical size of the sound horizon at last scattering, assuming that last scattering occurs much later than matter-radiation equality and that the speed of sound of the baryon-photon fluid is roughly $c_s^2 = 1/3$.

[You may assume the temperature at last scattering was approximately 1200 times the present temperature.]

(d) Relate the comoving time since last scattering, $\tau_0 - \tau_{LS} \simeq \tau_0$, to the comoving distance to the last scattering surface, r_{LS}. Calculate τ_0 and thus r_{LS}.

[*Hint: It is useful to work in terms of the variable* $\beta \equiv \sqrt{1-\Omega_m}$ *and you may use the mathematical expressions given at the end of the question.*]

(e) Infer the present angular size of the sound horizon at decoupling as a function of Ω_m. To which multipole moment, or ℓ value, does this correspond?

If the universe becomes reionized at an early enough redshift, z_{re}, then the CMB photons may have rescattered off of the free electrons. The optical depth for scattering is given by

$$\kappa = \int_{t(z_{re})}^{t_0} \sigma_T n_e c\, dt,$$

where n_e is the density of free electrons, and σ_T is the Thomson scattering cross section.

(f) Relate the optical depth to the redshift of reionization, the Hubble constant, and the baryon and matter densities for the case a matter dominated open universe; that is, find $\kappa(z_{re}, H_0, \Omega_b, \Omega_m)$. Assume that after reionization all the electrons are free.

(g) For a flat universe, estimate the reionization redshift that corresponds to $\kappa = 1$, in a $\Omega_b = 0.05$, $H_0 = 70\,\mathrm{km\,sec^{-1}\,Mpc^{-1}}$. How will this redshift change if the universe is significantly open? Explain your reasoning.

[*Note: the dimensionless combination*

$$\frac{\sigma_T c H_0}{4\pi G m_p} = 0.032\,,$$

where m_p *is the proton mass.*]

You may use the following relations:

$$\int_0^r \frac{da}{(1 + a^2/b^2)^{\frac{1}{2}}} = b \sinh^{-1}\left(\frac{r}{b}\right) ,$$

$$\int_0^r \frac{da}{(ba + a^2)^{\frac{1}{2}}} = 2 \ln\left[\sqrt{\frac{r}{b}} + \sqrt{1 + \frac{r}{b}}\right] ,$$

$$\int \frac{da}{a^2[a^2 + ba]^{\frac{1}{2}}} = \frac{2}{3b^2 a^{\frac{3}{2}}}(2a - b)(a + b)^{\frac{1}{2}} .$$

3 (a) A cosmological phase transition takes place in the Early Universe where a group G is spontaneously broken to H.

(i) Define the vacuum manifold M in terms of G and H,

(ii) Explain the significance of the homtopy groups $\pi_0(M)$, $\pi_1(M)$ and $\pi_2(M)$ in terms of the structure of the vacuum manifold and the types of the topological defects which would form during the phase transition.

(b) Consider spontaneous symmetry breaking in the Abelian-Higgs model,

$$\mathcal{L} = \overline{D_\mu \Phi} D^\mu \Phi - V(\Phi) - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,,$$

where the covariant derivative is $D_\mu = \partial_\mu - ieA_\mu$, the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the potential is

$$V(\Phi) = \frac{\lambda}{4}\left(\overline{\Phi}\Phi - \eta^2\right)^2 \,.$$

In this case what are G, H and M, and what type of topological defect would be formed in the corresponding phase transition. By writing Φ in terms of two real fields ϕ and ψ,

$$\Phi(\phi, \psi) = \left(\eta + \frac{\phi}{\sqrt{2}}\right) \exp\left[\frac{i\psi}{\eta\sqrt{2}}\right] ,$$

and the gauge field A_μ in terms of a modified field A'_μ and ψ

$$A_\mu = A'_\mu + \frac{\partial_\mu \psi}{e\eta\sqrt{2}} \,,$$

compute the particle spectrum of the model in the broken phase, paying particular attention to the number of degrees of freedom in the symmetric and broken phases.

(c) Consider a model where $SU(5)$ gauge symmetry is broken to the standard model of particle physics $SU(3)\times SU(2)\times U(1)$ at a 2nd order phase transition characterized by the energy scale η.

(i) What kind of defects would be formed in such a transition and what length and mass scales characterize their properties?

(ii) Assuming the Kibble causality bound is saturated, roughly estimate the initial density of defects, ρ_i, as well as the time of formation t_i. [*You may assume that there are* $\mathcal{N} \approx 100$ *relativistic degrees of freedom at* t_i.]

(iii) Explain how the density of the defects will evolve in terms of the scale factor $a(t)$. Assuming that these defects have a density which is less than 1% of the critical density today, deduce a bound on η if they interact very little after their formation. Comment on the implications of this result for grand unification in general. How might this be alleviated?

4

Consider a flat universe, which is composed of cold dark matter, radiation and some other component X, with equation of state $p = w\rho$ and densities relative to critical given by $\Omega_{\rm m}$, $\Omega_{\rm r}$ and $\Omega_{\rm X}$.

(a) The luminosity distance is defined as

$$d_L(z) = (1+z)\int_0^z \frac{dz'}{H(z')}\,,$$

and the deceleration parameter q_0 is defined in terms of it by

$$H_0 d_L = z + \tfrac{1}{2}(1-q_0)z^2 + {\rm O}(z^3)\,,$$

where H_0 is the present value of the Hubble constant.

Compute q_0 in terms of $\Omega_{\rm r}$, $\Omega_{\rm X}$ and w.

(b) Discuss the properties of the energy component X, required for it to make the standard cosmology compatible with recent obervations of the cosmic microwave background, type Ia supernovae and dynamical measurements of galaxy clusters. In particular, compare and contrast to those of a perfect fluid component.

(c) Now assume that the X-matter is in the form of a scalar field ϕ with potential $V(\phi)$. The Lagrangian of such a model is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,,$$

and the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\,.$$

(i) If the potential is

$$V(\phi) = \tfrac{1}{2}m^2\phi^2\,,$$

deduce the equation of motion for ϕ in conformal time and by comparing time scales make an order of magnitude estimate of the mass m and the value of ϕ today.

(ii) If the X-matter has an equation of state

$$p = -\frac{1}{3}\rho$$

deduce that

$$\phi' = \frac{2a^2\rho_\phi}{3}\,,$$

where $\rho_\phi = T_{00}$, and hence show that

$$a = A\cosh[B(\phi-\phi_0)+C] + D\,,$$

where A, B, C, D are constants to be determined. Therefore, using this compute the potential parameterized by the value of ϕ at the present day,

Ω_{m}, Ω_{X} and Ω_{r}. You may assume that $(\Omega_{\mathrm{m}}/2)^2 > \Omega_{\mathrm{r}}$, and do not use the relation $\Omega_{\mathrm{m}} + \Omega_{\mathrm{r}} + \Omega_{\mathrm{X}} = 1$.

You may use the following relation

$$\int dx \, \frac{1}{\sqrt{x^2 - 1}} = \cosh^{-1} x \, .$$