UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2001 1.30 to 4.30

PAPER 7

BANACH ALGEBRAS

*Attempt **TWO** questions from Section A, and **ONE** question from Section B.*
The questions carry equal weight.

All Banach algebras should be taken to be over the complex field.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Let A be a unital Banach algebra and let G be the group of all invertible elements of A. Prove that G is an open subset of A and also that the mapping $x \mapsto x^{-1}$ is a homeomorphism of G onto itself.

Define the *spectrum*, $\operatorname{Sp} x$ $(= \operatorname{Sp}_A x)$, of an element $x \in A$. Prove that $\operatorname{Sp} x$ is a non-empty, compact subset of $\mathbb{C}$.

Let B be a closed, unital subalgebra of A and let $x \in B$. Prove that $\mathbb{C} \setminus \operatorname{Sp}_B x$ is an open-and-closed subset of $\mathbb{C} \setminus \operatorname{Sp}_A x$.

[N.B. *you should not use any results concerning* $\partial \operatorname{Sp}_A x$*,* $\partial \operatorname{Sp}_B x$*, unless you first prove them.*]

Deduce that:

(i) if U is a component of $\mathbb{C} \setminus \operatorname{Sp}_A x$, then either U is also a component of $\mathbb{C} \setminus \operatorname{Sp}_B x$ or $U \subseteq \operatorname{Sp}_B x$;

(ii) $\mathbb{C} \setminus \operatorname{Sp}_B x$ and $\mathbb{C} \setminus \operatorname{Sp}_A x$ have the same unbounded component;

(iii) $\partial \operatorname{Sp}_B x \subseteq \partial \operatorname{Sp}_A x$.

2 Let A be a unital Banach algebra, let $x \in A$ and let

$$\exp x = \sum_{n \geqslant 0} \frac{x^n}{n!} .$$

Explain why the element $\exp x$ is well-defined.

Prove that if $x, y \in A$ and if $xy = yx$, then $\exp(x + y) = \exp x \, \exp y$. Deduce that, for every $x \in A$, $\exp x$ is an invertible element of A, with $(\exp x)^{-1} = \exp(-x)$.

Prove that if $u \in A$ and if 0 belongs to the unbounded component of $\mathbb{C} \setminus \operatorname{Sp}_A u$, then $u = \exp x$ for some element $x \in A$.

[*The holomorphic functional calculus theorem may be quoted without proof*.]

Let G be the group of invertible elements of A, topologized as a subset of A in its norm-topology, and let G_0 be the component of G that contains the identity element 1 of A. Prove that G_0 is an open-and-closed, normal subgroup of G, and that G_0 consists of all finite products of exponentials of elements of A.

[*You may assume, without proof, that G is an open subset of A and that the group operations are continuous*.]

3 Define an *involution* $x \mapsto x^*$ on a unital Banach algebra A. Prove that, if A is any Banach algebra with involution and $x \in A$, then $\mathrm{Sp}\, x^* = \{\overline{\lambda} : \lambda \in \mathrm{Sp}\, x\}$.

Define a (unital) C^*-*algebra* (in the abstract sense). Define what is meant by a *normal* element of such an algebra.

Let A be a unital C^*-algebra and let $x \in A$. Prove that:

(i) $\|x^*\| = \|x\|$;

(ii) if x is normal then $r(x) = \|x\|$;

(iii) if A is commutative and if φ is a character on A, then $\varphi(x^*) = \overline{\varphi(x)}$;

Let $p(z_1, z_2)$ be a complex polynomial in two variables, and let x be a normal element of the C^*-algebra A. Prove that:

$$\mathrm{Sp}\, p(x, x^*) = \{p(\lambda, \overline{\lambda}) : \lambda \in \mathrm{Sp}\, x\}\,.$$

SECTION B

You are reminded that you should answer only **ONE** *question from this section*

4 Let K be a non-empty, compact subset of $\mathbb{C}$. Let $C(K)$ be the usual commutative Banach algebra of all continuous, complex-valued functions on K, let $G(K)$ be the group of invertible elements of $C(K)$ and let $E(K) = \{\exp f : f \in C(K)\}$. Let $\mathbf{a} = \{a_1, a_2, a_3, \ldots\}$ be a (finite or infinite) sequence of points consisting of precisely one point from each bounded component of $\mathbb{C} \setminus K$. Let $G(\mathbf{a})$ be the subgroup of $G(K)$ consisting of (the restrictions to K of) all rational functions of the form

$$r(z) = (z - a_1)^{k_1}(z - a_2)^{k_2} \ldots\,,$$

where $k_i \in \mathbb{Z}$ and $k_i = 0$ for all sufficiently large i. (In the special case where $\mathbb{C} \setminus K$ is connected, let $G(\mathbf{a}) = \{1\}$.). Write an account to explain, in outline, how to prove that $G(K)$ is the internal direct product of groups, $G(K) = E(K) \times G(\mathbf{a})$.

Explain how this result may be used to prove the Jordan curve theorem.

5 Write an essay on the Borel functional calculus for a bounded normal operator T on a Hilbert space H.

[*You should assume, without proof, any relevant general results on Banach algebras, C^*-algebras and the topology of compact Hausdorff spaces.*]