UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2001 1.30 to 4.30

PAPER 68

GENERAL RELATIVITY

Attempt any ***THREE*** *questions. The questions are of equal weight.*

Candidates may make free use of the information given on the accompanying sheet.

Information

The signature is $(+\ -\ -\ -)$, *and the curvature tensor conventions are defined by*

$$R^i{}_{kmn} = \Gamma^i{}_{km,n} - \Gamma^i{}_{kn,m} - \Gamma^i{}_{pm}\Gamma^p{}_{kn} + \Gamma^i{}_{pn}\Gamma^p{}_{km}\,.$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the concept of a *linear connection* ∇ on a manifold $\mathcal{M}$. If $\mathcal{M}$ possesses a metric tensor g whose components with respect to a local coordinate system are g_{ab}, show that there exists a unique symmetric *Levi-Civita* or *metric* connection such that $\nabla g = 0$, and determine its components $\Gamma^a{}_{bc}$.

Next suppose that $\overline{g}_{ab}$ are the components of another symmetric tensor $\overline{g}$ on $\mathcal{M}$, and let $\overline{\Gamma}{}^a{}_{bc}$ be the components of that symmetric connection for which $\overline{\nabla}\,\overline{g} = 0$. Show that $S^a{}_{bc} = \overline{\Gamma}{}^a{}_{bc} - \Gamma^a{}_{bc}$ are the components of a tensor S.

Finally suppose that the connections Γ and $\overline{\Gamma}$ have the same geodesics. Show that there exists a covector V_c (to be determined) such that

$$S^a{}_{bc} = 2\delta^a{}_{(b}V_{c)}.$$

2 Write an essay on gauge-invariant, linearized, vacuum perturbations of flat Minkowski spacetimes.

[*You may use any information from the lecture handout included with this examination paper.*]

3 Using standard notation the action S for the *Brans-Dicke* theory of gravity is given by

$$16\pi G S = \int \sqrt{-g}\left[R\Phi + \frac{\omega\Phi_{,a}\Phi^{,a}}{\Phi} + 16\pi G L_{\text{matter}}\right] d\Omega,$$

where Φ is a scalar field, ω is a coupling constant and L_{matter} is the Lagrangian of the matter content. Derive the field equation

$$\Phi G^{ab} + (g^{ac}g^{bd} - g^{ab}g^{cd})\Phi_{;cd} + \omega\Phi^{-1}(g^{ac}g^{bd} - \tfrac{1}{2}g^{ab}g^{cd})\Phi_{,c}\Phi_{,d} = -8\pi G T^{ab}_{\text{matter}},$$

where G^{ab} is the Einstein tensor, and show that the field equation for Φ can be written in the form

$$\Box\Phi = \frac{8\pi G}{3+2\omega}\, g_{cd}\, T^{cd}_{\text{matter}}.$$

[*You may use freely the following results for variations:*

(a) $\delta g^{cd} = -g^{ac}g^{bd}\delta g_{ab}$,

(b) $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}$,

(c) $\delta R_{bc} = \left(\delta\Gamma^a{}_{ba}\right)_{;c} - \left(\delta\Gamma^a{}_{bc}\right)_{;a}$.]

4 Toy ($1+1$-dimensional) models with line elements for *de Sitter spacetime*

$$ds^2 = dt^2 - \cosh^2 t \, d\chi^2,$$

and *anti-de Sitter spacetime*

$$ds^2 = \cosh^2 r \, dt^2 - dr^2,$$

were introduced in the lectures. What are the ranges of the various coordinates?

Write an essay on the similarities and differences between these two models, paying particular attention to the geodesics, conformal structure and horizons.