UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2001 9 to 11

PAPER 67

COSMOLOGY

*Attempt **THREE** questions. The questions are of equal weight.*

Candidates may make free use of the information given on the accompanying sheet.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose that an FRW universe is filled with bright point sources with a uniform number density $n(t)$ each with a constant absolute luminosity L (i.e. emitting a time-independent power).

(i) By considering the proper volume element $a^3(t)\left({}^{(3)}g\right)^{1/2} dr\, d\theta\, d\phi$ [*see the Information sheet*], show that the number of sources dN observed today at t_0 whose light was emitted between $t - dt$ and t is

$$dN = 4\pi a^2(t) r^2(t) n(t) dt\,.$$

(ii) Briefly explain the physical origin of the main terms in the apparent luminosity formula

$$F = \frac{L}{4\pi d_L^2} = \frac{L}{4\pi a^2(t_0) r^2 (1+z)^2}\,,$$

where d_L is the luminosity distance. Calculate the total power observed today from all the uniform sources (described above) in a flat matter-dominated universe ($k = \Lambda = 0$) with the solution $a = (t/t_0)^{2/3}$.

(iii) A matter-dominated open universe ($k < 0, \Lambda = 0$) has the parametric solution [*Do not derive this result*]

$$a(\tau) = \frac{\Omega_0}{2(1-\Omega_0)}[\cosh\sqrt{-k}\tau - 1], \quad t(\tau) = \frac{H_0^{-1}}{2}\frac{\Omega_0}{(1-\Omega_o)^{\frac{3}{2}}}[\sinh\sqrt{-k}\tau - \sqrt{-k}\tau],$$

where $-k = H_0^2(1-\Omega_0)$ with the Hubble parameter today H_0 and the relative matter density Ω_0. Find the total power observed today from the uniform sources described above. [*Leave your result in parametric form.*] Hence, or otherwise, compare the flat universe result with an open universe in the limit $\Omega_0 \to 0$.

2 We wish to describe the synthesis of light elements in the early universe using the appropriate results from their equilibrium distributions given on the Information Sheet.

(i) The interaction $\nu_e + n \leftrightarrow p + e^-$, has the rate $\Gamma \approx G_F^2 T^5$ where the Fermi constant $G_F \approx 10^{-5} m_p^{-2}$ and $m_p \approx 1\,\text{GeV}$. Roughly estimate the decoupling temperature T_d for neutrinos ν_e for this interaction, given that the effective number of degrees of freedom $\mathcal{N}$ at that time is $\mathcal{N} = 10.75 \approx (1.5)^6$. Explain why the ratio of relative neutron and proton densities at neutrino decoupling is given simply by

$$\frac{n_n}{n_p} = \frac{X_n}{X_p} = \exp(-Q/T_d)\,,$$

where $Q = m_n - m_p$ and $X_i \equiv n_i/n_B$ with n_B the total baryon number of the universe. (Here, you may assume $\mu_n \approx \mu_p$ are both negligible.)

(ii) Subsequently, deuterium forms through the interaction $p + n \leftrightarrow D + \gamma$. For the fractional deuterium abundance X_D, find an expression for the ratio

$$\frac{X_D}{X_n X_p}$$

in terms of the binding energy $B_D = m_D - m_n - m_p$, the baryon-to-photon ratio $\eta \equiv n_B/n_\gamma$ and the temperature T. (Assume that $g_D = 3$.)

(iii) Why is the relative density of deuterium X_D very sensitive to the baryon number of the universe n_B, while the relative mass fraction of helium-4 Y_P is not?

3 Suppose a general FRW universe ($k \neq 0$) contains matter density ρ_M, radiation density ρ_R, and a cosmological constant Λ.

(i) Show that the Friedmann equation [*see the Information sheet*] can be rewritten as

$$H^2 = H_0^2 \left[\Omega_{R_0} a^{-4} + \Omega_{M_0} a^{-3} + \Omega_{\Lambda_0} + (1-\Omega_0) a^{-2}\right]\,,$$

where H is the Hubble parameter $H = \dot{a}/a$ and the relative density parameters Ω_{M_0}, Ω_{R_0}, Ω_{Λ_0}, and Ω_0 should be defined today at t_0 in terms of ρ_M, ρ_R, Λ and k (with $a(t_0) = 1$).

(ii) For a $\Lambda = 0$ model, find a solution for the total density parameter Ω as a function of the scalefactor a, given Ω_{M_0}, Ω_{R_0} and Ω_0 today. Hence, briefly explain the flatness problem of the standard cosmology.

(iii) For an empty open universe ($\Omega_M = \Omega_R = 0$) with $k < 0$ and $\Lambda > 0$, find an appropriately normalised solution for the scalefactor $a(t)$.

4 In a flat FRW universe filled with cold dark matter (CDM) and radiation, the density perturbations obey the coupled equations [*Do not attempt to derive these.*]

$$\delta_c'' + \frac{a'}{a}\delta_c' - \frac{3}{2}\left(\frac{a'}{a}\right)^2 [\Omega_c\delta_c + 2\Omega_r\delta_r] = 0\,,$$
$$\delta_r'' + \tfrac{1}{3}k^2\delta_r - \tfrac{4}{3}\delta_c'' = 0\,,$$

where δ_c and δ_r are the CDM and radiation perturbations respectively, Ω_c and Ω_r are the CDM and fractional densities respectively, $k = |\mathbf{k}|$ is the comoving wavenumber, and primes $'$ denote derivatives with respect to conformal time τ ($dt = a d\tau$).

(i) For adiabatic perturbations on superhorizon scales ($k\tau \ll 2\pi$), the number of photons per CDM particle remains fixed. Explain why this implies $\delta_r = \frac{4}{3}\delta_c$ in this case?

(ii) Deep in the matter era (i.e. assume $\Omega_c \approx 1$ and $\tau \gg \tau_{eq}$), show that the general CDM solution takes the approximate form

$$\delta_c(\mathbf{k},\tau) \approx A(\mathbf{k})\Big(\frac{\tau}{\tau_i}\Big)^2 + B(\mathbf{k})\Big(\frac{\tau}{\tau_i}\Big)^{-3}, \qquad (A,\ B \text{ arbitrary})\,,$$

and is valid on both subhorizon and superhorizon scales. Find an approximate form for the subhorizon ($k\tau \gg 2\pi$) radiation perturbation δ_r in this regime. [*You need not match your solution at $k\tau = 2\pi$.*] Sketch the behaviour of both δ_c and δ_r as a function of τ as they cross the horizon.

(iii) Deep in the radiation era (i.e. assume $\Omega_r \approx 1$ and $\tau \ll \tau_{eq}$), find the general solution for the CDM perturbation δ_c on superhorizon scales and the δ_c solution on subhorizon scales. Clearly state the approximations you make.

5 In the comoving synchronous gauge the perturbed FRW metric ($k = 0$) takes the form,

$$ds^2 = a^2(\tau)\left[d\tau^2 - (\delta_{ij} - h_{ij})dx^i dx^j\right] = dt^2 - d\mathbf{r}^2\,.$$

(i) Consider a photon trajectory in the direction $\hat{\mathbf{n}}$ and show that the velocity variation between two comoving points (separated by a proper distance δr) along the photon trajectory $\delta t \approx \delta r$ is given by

$$\Delta v = \frac{d\,\delta r}{dt} \approx \left[\frac{\dot{a}}{a} - \frac{1}{2}\dot{h}_{ij}\hat{n}_i\hat{n}_j\right]\delta t\,.$$

Thus relate temperature fluctuations in the photon background to time variations in the metric perturbation h_{ij},

$$\frac{\delta T}{T} \approx \frac{\delta\nu}{\nu} \approx \frac{1}{2}\int_{t_{\rm dec}}^{t_0} \dot{h}_{ij}\hat{n}_i\hat{n}_j\,dt\,. \qquad (\dagger)$$

(ii) In Fourier space, the scalar metric perturbation h_{ij} can be decomposed about the wave vector direction $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ such that the Sachs-Wolfe integral (†) becomes [*Do not derive this result.*]

$$\frac{\delta T}{T} = \frac{1}{2}\int_{\tau_{\rm dec}}^{\tau_0} d\tau \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\hat{\mathbf{n}}\tau}\left(\frac{1}{3}h'\delta_{ij} + (\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})h'_S\right)\hat{n}_i\hat{n}_j\ ,$$

where $h = h_{ii}$ is the trace, h_S is the anisotropic scalar and τ is conformal time ($dt = a d\tau$). Given that $h - h_S = 0$ and $h = A\tau^2k^2$ in the matter era ($\Lambda = 0$), show that this integral reduces to

$$\frac{\delta T}{T} = -\left[\sum_{\mathbf{k}} i\mathbf{k}\cdot\hat{\mathbf{n}}\,A\tau\,e^{i\mathbf{k}\cdot\hat{\mathbf{n}}\tau}\right]_{\tau_{\rm dec}}^{\tau_0} + \left[\sum_{\mathbf{k}} Ae^{i\mathbf{k}\cdot\hat{\mathbf{n}}\tau}\right]_{\tau_{\rm dec}}^{\tau_0}\,.$$

Briefly discuss the scale-dependence of these temperature fluctuations on large angular scales.