UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 1.30 to 4.30

PAPER 63

THE STANDARD MODEL

*Attempt **THREE** questions.*

The questions are of equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For $\phi = (\phi_1, \ldots, \phi_n)$ a real multi-component scalar field, then assume a potential $V(\phi)$ is invariant under the action of a continuous group$G$, so that for any $g \in G$, $V(g\phi) = V(\phi)$. Suppose $\phi_0 \neq 0$ is a particular ϕ for which V has its minimum. Describe how the unbroken subgroup$H \subset G$ may be defined and explain how there are in general $\dim G - \dim H$ massless modes.

[You may assume that if $V({\phi_0}') = V(\phi_0)$ then${\phi_0}' = g\phi_0$ for some $g \in G$.]

If G is now a local gauge group, with $A_{\mu a}$, $a = 1, \ldots, \dim G$, the corresponding gauge fields,the associated Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(D^\mu \phi)^T D_\mu \phi - \tfrac{1}{4} F^{\mu\nu}{}_a F_{\mu\nu a} - V(\phi)\,,$$

where $F_{\mu\nu a}$ is the associated field strength, $D_\mu \phi = \partial_\mu \phi + g A_{\mu a} T_a \phi$ with T_a matrix generators of G, ${T_a}^T = -T_a$. Explain why we may impose the condition $\phi^T T_a \phi_0 = 0$. Show how the gauge fields have masses defined by the matrix $g^2 (T_a \phi_0)^T T_b \phi_0$ and that $\dim G - \dim H$ gauge fields have a non zero mass. Verify also that there are now no massless scalar fields.Briefly describe the relevance of these considerations to the standard model.

2 In the standard model the interaction of the neutral vector Z with fermion fields ψ is described by

$$\mathcal{L}_I = \frac{g}{2\cos\theta_W} J_n^\mu Z_\mu \,,$$

where the neutral current is given by

$$J_n^\mu = \overline{\psi}\gamma^\mu\big((1-\gamma_5)T_3 - 2\sin^2\theta_W Q\big)\psi \,,$$

with $\mathbf{T}$ the generators of $SU(2)_T$ acting on ψ and Q the associated charge matrix. For the leptonic decay $Z \to \ell\bar{\ell}$ show that $J_n^\mu = \bar{\ell}\gamma^\mu(c_V - c_A\gamma_5)\ell$ where for $\ell = e$,$c_V = 2\sin^2\theta_W - \frac{1}{2}$, $c_A = -\frac{1}{2}$ and for $\ell = \nu_e$, $c_V = c_A = \frac{1}{2}$.Assuming

$$\langle 0|Z_\mu(0)|Z(p)\rangle = \epsilon_\mu(p)\,, \qquad \sum_{Z\text{ spins}} \epsilon_\mu(p)\epsilon_\nu(p)^* = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}\,,$$

show that for $\mathcal{M}$ the amplitude for $Z \to \ell\bar{\ell}$ and neglecting any lepton masses,$\sum_{\text{spins}} |\mathcal{M}|^2 = g^2({c_V}^2 + {c_A}^2)m_Z^2/\cos^2\theta_W$.Hence obtain, for $G_F/\sqrt{2} = g^2/(8m_W^2)$,

$$\Gamma_{Z\to\ell\bar{\ell}} = \frac{G_F}{\sqrt{2}}\,\frac{m_Z^3}{6\pi}({c_V}^2 + {c_A}^2)\,.$$

[The formula for the decay width of a particle with mass m is

$$\Gamma = \frac{1}{2m}\sum_X (2\pi)^4\delta^4(p-p_X)\,|\langle X|\mathcal{L}_I|p\rangle|^2\,, \qquad \sum_X = \prod_{\text{momenta}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \sum_{\text{spins}}\,.$$

You may also use $\mathrm{tr}(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = 4(g_{\alpha\beta}\,g_{\gamma\delta} + g_{\alpha\delta}\,g_{\beta\gamma} - g_{\alpha\gamma}\,g_{\beta\delta})$.]

3 For a hadron H of momentum P, $P^2 = M^2$, represented by the state $|P\rangle$ define

$$W^{\mu\nu}(q,P) = \frac{1}{4\pi}\sum_X (2\pi)^4\delta^4(P-p_X-q)\,\langle P|J^\mu|X\rangle\langle X|J^\nu|P\rangle$$
$$= \Big(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\Big)W_1 + \Big(P^\mu - \frac{P{\cdot}q}{q^2}\,q^\mu\Big)\Big(P^\nu - \frac{P{\cdot}q}{q^2}\,q^\nu\Big)W_2\,,$$

where $J^\mu = \sum_f Q_f \bar{q}_f\gamma^\mu q_f$ is the electromagnetic current.If $\nu = P{\cdot}q$, $x = -q^2/2\nu$ and for $W_1 = F_1(x,-q^2)$, $\nu W_2 = F_2(x,-q^2)$ show that as $-q^2\to\infty$ with suitable assumptions,

$$F_1(x,-q^2) \sim \tfrac{1}{2}\sum_f Q_f^2\big(q_f(x)+\bar{q}_f(x)\big)\,,\quad F_2(x,-q^2)\sim x\sum_f Q_f^2\big(q_f(x)+\bar{q}_f(x)\big)\,.$$

Explain briefly why we may expect

$$\int_0^1 \mathrm{d}x\,\big(q_f(x)-\bar{q}_f(x)\big) = N_f\,,$$

where N_f is the number of quarks of type f in the hadron H.If only u, d quarks are relevant, so that $\bar{q}_f = 0$, and F_2^{proton},F_2^{neutron} are the functions for H corresponding to a proton, neutron what are the values of the integrals

$$\int_0^1 \frac{\mathrm{d}x}{x}\,F_2^{\text{proton}}(x,-q^2)\,,\qquad \int_0^1 \frac{\mathrm{d}x}{x}\,F_2^{\text{neutron}}(x,-q^2)\,,$$

as $-q^2\to\infty$.

[$\gamma^\mu\gamma^\lambda\gamma^\nu = g^{\mu\lambda}\gamma^\nu + g^{\nu\lambda}\gamma^\mu - g^{\mu\nu}\gamma^\lambda + i\epsilon^{\mu\nu\lambda\kappa}\gamma_\kappa\gamma_5$.]

4 What are the quantum numbers of the quarks and leptons for one generation in the standard model under the gauge group $SU(3)_{\text{colour}}\times SU(2)_T\times U(1)_Y$? Show that if $Q = T_3 + Y$, where $\mathbf{T}$ are the generators of $SU(2)_T$ and Y is the generator of $U(1)_Y$, then the eigenvalues of Q give the correct charges for the quarks and leptons. Due to anomalies the following constraints must be imposed,

$$\mathrm{tr}_L\big(T_3{}^2Y\big) - \mathrm{tr}_R\big(T_3{}^2Y\big) = 0\,,\qquad \mathrm{tr}_L\big(Y^3\big)-\mathrm{tr}_R\big(Y^3\big) = 0\,.$$

where tr_L, tr_R denote the traces for the generators acting on left handed, right handed fermion fields respectively. Show that, for the usual representations of $SU(2)_T\times U(1)_Y$, $\mathrm{tr}_R\big(T_3{}^2Y\big) = 0$ and also that $\mathrm{tr}_L\big(T_3{}^2Y\big) = \frac{1}{4}\,\mathrm{tr}_L\big(Y\big) = \frac{1}{4}\,\mathrm{tr}_L\big(Q\big)$. Hence verify that the first condition is satisfied for each generation independently. Show further that the second condition holds.

5 Show how QCD for $m_u = m_d = 0$ has a chiral $SU(2) \times SU(2)$ symmetry where if $q = \begin{pmatrix} u \\ d \end{pmatrix}$ then $q_R \to A q_R,\ q_L \to B q_L$ for $A, B \in SU(2)$. Let $V_{ij} = \overline{q_{Lj}} q_{Ri}$. How does this transform? If

$$\langle 0 | V_{ij} | 0 \rangle = -v\, \delta_{ij}\,, \quad v \text{ real},\ v > 0\,,$$

what is the symmetry reduced to? Describe in outline how this leads to massless pions. Assume that at low energies the pion fields $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ determine a matrix $U(\boldsymbol{\pi}) = \exp(i\boldsymbol{\pi}\cdot\boldsymbol{\tau}/F) \in SU(2)$ which transforms under a chiral transformation as $U(\boldsymbol{\pi}) \to AU(\boldsymbol{\pi})B^{-1}$. Show that, for up to two derivatives, there is then a unique Lagrangian for massless pions, invariant under chiral $SU(2) \times SU(2)$ symmetry, of the form

$$\mathcal{L}_\pi = \tfrac{1}{4}F^2 \mathrm{tr}\big(\partial^\mu U(\boldsymbol{\pi})^\dagger \partial_\mu U(\boldsymbol{\pi})\big) = \tfrac{1}{2}\,\partial^\mu \boldsymbol{\pi}\cdot\partial_\mu \boldsymbol{\pi} + \dots\,.$$

In QCD show that a mass term for the u, d quarks may be introduced by

$$\mathcal{L}_m = -\mathrm{tr}\big(\mathcal{M}(V + V^\dagger)\big)\,, \quad \mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \quad m_u, m_d > 0\,.$$

An equivalent Lagrangian $\mathcal{L}_{\pi,m}$ for the pion fields may be constructed by letting $V \to -vU(\boldsymbol{\pi})$. By expanding $\mathcal{L}_{\pi,m}$to second order determine the expected mass for the pions.