UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2001 1.30 to 4.30

PAPER 6

LINEAR ANALYSIS

*Answer **FOUR** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose that V and W are Banach spaces and that T is a bounded linear mapping from V into W with the property that there exist $R > 0$ and $0 < k < 1$ such that if $y \in W$ then there exists $x \in V$ with $\|x\| \leqslant R\|y\|$ and $\|T(x) - y\| \leqslant k\|y\|$.

Show that if $y \in W$ there exists $x \in V$ with $\|x\| \leqslant R\|y\|/(1-k)$ and $T(x) = y$.

Suppose that g is a bounded continuous real-valued function on a closed subset Y of a metric space (X, d). Show that there is a bounded continuous real-valued function h on X which extends g, and with

$$\sup\{|h(x)| : x \in X\} = \sup\{|g(y)| : y \in Y\}.$$

Show that an unbounded continuous function f on Y has a continuous extension to X. [Hint: Consider $g(y) = \tanh f(y)$: if h is an extension of g, then Y and $\{x : |h(x)| \geqslant 1\}$ are disjoint.]

2 Suppose that ϕ is a convex function on a real vector space V and that f is a linear functional on a subspace W of V which satisfies $f(x) \leqslant \phi(x)$ for all $x \in W$. Show that there exists a linear functional g on V such that $g(x) = f(x)$ for $x \in W$ and $g(y) \leqslant \phi(y)$ for $y \in V$.

By considering functions of the form $\varphi_y(z) = \phi(y+z) - \phi(y)$, show that there exists a set A of affine functions on V such that

$$\phi(y) = \sup\{a(y) : a \in A\}$$

for each $y \in V$.

[An affine function a on a vector space is a function of the form $a = \ell + c$, where ℓ is a linear functional and c is a constant.]

3 Explain how to construct the Stone-Čech compactification βX of a completely regular Hausdorff topological space X, and establish the fundamental properties of Stone-Čech compactifications.

Show that if X is a space with the discrete topology and A and B are disjoint closed subsets of βX then there is a splitting $\beta X = U \cup V$ with $A \subseteq U$ and $B \subseteq V$.

Give an example of a separable compact Hausdorff topological space K for which the Banach space $C(K)$ is not separable.

4 Define the **weak topology** on a Banach space E. Show that a sequence (x_n) in a weakly compact subset K of E has a subsequence which is weakly convergent to a point of K.

Suppose that T is a bounded linear operator from E into a Banach space F. Show that the following are equivalent:

(i) if $x_n \to 0$ weakly in E then $T(x_n) \to 0$ in norm in F;

(ii) if K is weakly compact in E then $T(K)$ is norm compact in F.

Show that the inclusion mapping j_2 of $C(\mathbb{T})$ into $L^2(\mathbb{T}, \mu)$ satisfies these conditions (where μ is normalised Lebesgue measure on $\mathbb{T} = [0, 2\pi]$). Is j_2 compact? Justify your answer.

5 Show that a bounded linear operator between Hilbert spaces H_1 and H_2 is Hilbert-Schmidt if and only if it is 2-summing, with equality of norms. [You may assume basic properties of Hilbert-Schmidt operators, such as the tensorial representation.]

Show that the 2-summing norm of the identity mapping of an n-dimensional normed space E is $\sqrt{n}$.

Suppose that E is a linear subspace of a Banach space X. Show that there is a projection P of X onto E with $\|P\| \leqslant \sqrt{n}$.