UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2001 9 to 12

PAPER 58

APPROXIMATION THEORY

Attempt ***THREE*** *questions from Section A and at most* ***ONE*** *question from Section B.*

The questions attempted from Section B will carry twice the maximum mark of those attempted from Section A.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Let $\mathcal{S}$ be the spline space of degree 3 defined on the knot sequence

$$\delta = (t_1 = t_2 = t_3 = t_4 = -1 < t_5 = 0 < 1 = t_6 = t_7 = t_8 = t_9)$$

and let

$$f(x) = \begin{cases} \frac{27}{2}x^3 - \frac{27}{2}x^2 + 1, & x \in [0,1], \\ f(-x), & x \in [-1,0]. \end{cases}$$

The formula for the dual functionals $(\lambda_{j,k})$ to the B-spline basis $(N_{j,k})$

$$\lambda_{j,k}(g) = \frac{1}{(k-1)!}\sum_{\nu=1}^{k}(-1)^{\nu-1}\psi_{j,k}^{(\nu-1)}(\tau)g^{(k-\nu)}(\tau), \quad \lambda_{j,k}(N_{i,k}) = \delta_{ij},$$

where $\psi_{j,k} = (t_{j+1} - x)\ldots(t_{j+k-1} - x)$, and τ is any point in $[t_j, t_{j+k}]$, can be used to find the coefficients of the B-spline expansion of f as an element of $\mathcal{S}$. Hence show that f has the expansion

$$f = N_{1,4} - \frac{7}{2}N_{2,4} + \frac{11}{2}N_{3,4} - \frac{7}{2}N_{4,4} + N_{5,4}.$$

(Use symmetry $f(x) = f(-x)$ to halve the calculations.)

Define the condition number of $\mathcal{S}$, and deduce from this choice of f that it satisfies

$$\kappa(\mathcal{S}) \geq \frac{11}{2}.$$

2 Let $\mathcal{S}_{k,\delta}[0,1]$ be the space of splines of degree $k-1$ on a knot sequence $\delta = (t_j)_{j=1}^{n+k} \subset [0,1]$ such that $t_j < t_{j+k}$, and let $x = (x_i)_{i=1}^{n}$ be interpolation points obeying the conditions

$$N_{i,k}(x_i) \neq 0.$$

Let $P_x : C[0,1] \to \mathcal{S}_{k,\delta}$ be the map which associates with any $g \in C[0,1]$ the spline $P_x(g)$ from $\mathcal{S}$ which interpolates g at (x_i). Prove that

$$\|P_x\|_{L_\infty} \leq \|A_x^{-1}\|_{l_\infty} \qquad (*)$$

where A_x is the matrix $(N_{j,k}(x_i))_{i,j=1}^{n}$.

Compute $\|A_x^{-1}\|_{l_\infty}$ for $k = 3$, the Bernstein knot-sequence

$$\delta = (t_1 = t_2 = t_3 = 0 < 1 = t_4 = t_5 = t_6)$$

and the interpolation points

$$x_1 = 0, \quad x_2 = 1/2, \quad x_3 = 1.$$

Recall that, for the Bernstein knot-sequence (i.e., without interior knots), splines are just polynomials, so P_x is the quadratic polynomial interpolation projector. Find the norm $\|P_x\|$ directly and deduce that the bound $(*)$ is not sharp. (You may use the fact that the norm of polynomial interpolation projector is the maximal value among the norms of polynomials which take the values ± 1 at (x_i).)

3 State the Korovkin theorem.

Let $\mathcal{S}_{k,\delta_n}[0,1]$ be a sequence of spline spaces of degree $k-1$ with the knot-sequences

$$\delta_n = \{t_1^{(n)} = \ldots = t_k^{(n)} = 0 < t_{k+1}^{(n)} \leq \ldots \leq t_n^{(n)} < t_{n+1}^{(n)} = \ldots = t_{n+k}^{(n)} = 0\}$$

such that $|\delta_n| := \max_j |t_{j+1}^{(n)} - t_j^{(n)}| \to 0 \quad (n \to \infty)$. Consider the Schoenberg-type operator

$$V_n : C[0,1] \to \mathcal{S}_{k,\delta_n}[0,1], \quad V_n(g) = \sum_{j=1}^{n} g(\tau_j^{(n)}) N_{j,k,\delta_n}$$

with (N_{j,k,δ_n}) the B-spline basis for $\mathcal{S}_{k,\delta_n}$ and $\tau_j^{(n)}$ any point satisfying

$$t_j^{(n)} < \tau_j^{(n)} < t_{j+k}^{(n)}.$$

Using the Korovkin theorem prove that, with $k \geqslant 3$, for any $g \in C[0,1]$,

$$\|V_n g - g\|_{C[0,1]} \to 0 \quad (n \to \infty).$$

Hint. The polynomial x^l for $l = 0, 1$ and 2 can be expanded as $\sum_{j=1}^{n} a_{l,j} N_{j,k}(x)$, $0 \leq x \leq 1$, where (supressing the indices n and δ_n)

$$a_{0,j} = 1, \quad a_{1,j} = (k-1)^{-1} \sum_{l=j+1}^{j+k-1} t_l, \quad a_{2,j} = \binom{k-1}{2}^{-1} \sum_{j+1 \leq l < m \leq j+k-1} t_l t_m.$$

4 Prove the following variation of the Stone–Weierstrass theorem.

Suppose that the family $\mathcal{A}$ of real continuous functions $a(x)$ defined on a compact set T has the property that, for any two functions $a_1, a_2 \in \mathcal{A}$,

$$\max\{a_1, a_2\} \textit{ and } \min\{a_1, a_2\} \textit{ also belong to } \mathcal{A}.$$

If a function $f \in C(T)$ is such that for any two points $s, t \in T$ and for any $\epsilon > 0$ there is a function $a \in \mathcal{A}$ such that

$$|f(x) - a(x)| < \epsilon \textit{ for } x = s \textit{ and for } x = t,$$

then, on T, this f admits uniform approximation by functions $a \in \mathcal{A}$.

Derive as a corollary the following statement:

The set of all continuous piecewise linear functions is dense in the space of all continuous functions on $[0,1]$.

5 For periodic functions $f \in C(\mathbb{T})$, let $\sigma_{n-1}(f)$ be the Fejer operator

$$\sigma_{n-1}(f,x) = \int_{-\pi}^{\pi} F_{n-1}(t) f(x-t)dt, \quad F_{n-1}(t) := \frac{1}{2n}\frac{\sin^2 \frac{n}{2}t}{\sin^2 \frac{1}{2}t}, \quad \int_{-\pi}^{\pi} F_{n-1}(t)dt = 1,$$

which associates with each f the average of its partial Fourier sums up to the $(n-1)$-st order.

Prove the following: for $f \in C(\mathbb{T})$

$$\|\sigma_{n-1}(f) - f\|_{C(\mathbb{T})} \le \text{const} \cdot \omega(f, \frac{1}{\sqrt{n}}),$$

where $\omega(f,\delta) := \sup_{|x-y|<\delta} |f(x)-f(y)|$ is the (first) modulus of continuity of f.

Hint. Sometimes the integral of a function over $[0,\pi]$ can be estimated by splitting $\int_0^\pi = \int_0^\delta + \int_\delta^\pi$, estimating each term and choosing δ to minimize the sum.

SECTION B

6 Write an essay on the Weierstrass theorem. You should present Korovkin, Lebegues and Stone's theorems along with an outline of their proofs, and their corollaries.

7 Write an essay on spline interpolation. You should pay attention to the following issues: existence and uniqueness, the total positivity of the spline collocation matrix, the optimal interpolation set, and, perhaps, the condition number.