UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 1.30 to 4.30

PAPER 57

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Attempt ***THREE*** *questions from Section A and attempt* ***ONE*** *question from Section B.*

Each question from Section A carries the weight of 18 points and each question from Section B carries the weight of 36 points.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Prove that any ν-stage Runge-Kutta method of order 2ν is necessarily A-stable. [*You may use, without proof, properties of Padé approximants to the exponential.*]

2 The parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \kappa u$$

is given for $0 \leqslant x \leqslant 1, t \geqslant 0$, together with an initial condition at $t = 0$ and zero boundary conditions at $x = 0$ and $x = 1$.

(a) Prove (by separation of variables or otherwise) that the exact solution of the equation tends to zero as $t \to \infty$ for every intial condition if and only if $\kappa < \pi^2$.

(b) The equation is semidiscretized into

$$u_m' = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \kappa u_m, \qquad m = 1, 2, \ldots, M,$$

where $(M+1)\Delta x = 1$. Find the necessary and sufficient condition on κ that ensures $\lim_{t\to\infty}(t) = 0, m = 1, 2, \ldots, M$, for all possible initial conditions.

3 Given $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, we let $\mathbf{g}(\mathbf{y}) := \mathbf{y}'' = (\partial \mathbf{f}(\mathbf{y})/\partial \mathbf{y})\mathbf{f}(\mathbf{y})$. Determine the order of the two-step two-derivative method

$$\mathbf{y}_{n+1} - \frac{7}{15}h\mathbf{f}(\mathbf{y}_{n+1}) + \frac{1}{15}h^2\mathbf{g}(\mathbf{y}_{n+1}) = \mathbf{y}_{n-1} + h(\frac{16}{15}\mathbf{f}(\mathbf{y}_n) + \frac{7}{15}\mathbf{f}(\mathbf{y}_{n-1})) + \frac{1}{15}h^2\mathbf{g}(\mathbf{y}_{n-1}).$$

4 Prove that any positive-definite 9×9 matrix with the sparsity structure

$$\begin{bmatrix} \times & 0 & \times & \times & 0 & 0 & \times & 0 & 0 \\ 0 & \times & 0 & 0 & 0 & 0 & 0 & \times & 0 \\ \times & 0 & \times & 0 & 0 & 0 & 0 & 0 & 0 \\ \times & 0 & 0 & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \times & \times & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\ \times & 0 & 0 & 0 & 0 & \times & \times & 0 & \times \\ 0 & \times & 0 & 0 & \times & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \times & 0 & \times \end{bmatrix}$$

admits, after re-ordering, Guassian elimination without fill-in. Describe the procedure that yo uhave used in the proof and carefully state all relevant theorems.

5 The semi-discretization

$$\begin{aligned} u_m' &= \frac{1}{\Delta x}(-\frac{3}{2}u_m + 2u_{m+1} - \frac{1}{2}u_{m+2}), \quad m = 1, 2, \ldots, M-2, \Delta x = \frac{1}{M}, \\ u_{M-1}' &= \frac{1}{\Delta x}(-\frac{3}{2}u_{M-1} + 2u_M - \frac{1}{2}u_1), \\ u_M' &= \frac{1}{\Delta x}(-\frac{3}{2}u_M + 2u_1 - \frac{1}{2}u_2) \end{aligned}$$

approximates the solution of the advection equation $u_t = u_x$, given with intial conditions $u(x, 0) = \varphi(x), 0 \leqslant x \leqslant 1$, and *periodic* boundary conditions $u(0, t) = u(1, t) t \geqslant 0$. Prove that the method is stable. [*Hint: You might formulate the problem in matrix form,* $\mathbf{u}' = \frac{1}{\Delta x} A\mathbf{u}$*, say, and show that all eigenvectors of* A *are of the form* $[1, \omega_\ell, \ldots, \omega_\ell^{M-1}]^\top$ *for some complex numbers* $\{\omega_\ell : \ell = 1, 2, \ldots, M\}$.]

6 Determine the range of the real parameter α such that the multistep method

$$\begin{aligned} &\mathbf{y}_{n+3} - (1+2\alpha)\mathbf{y}_{n+2} + (1+2\alpha)\mathbf{y}_{n+1} - \mathbf{y}_n \\ =&\frac{1}{6}h[(5+\alpha)\mathbf{f}(\mathbf{y}_{n+3}) - (4+8\alpha)\mathbf{f}(\mathbf{y}_{n+2} + (11-5\alpha)\mathbf{f}(\mathbf{y}_{n+1})] \end{aligned}$$

is convergent.

What is the order of the method for different values of α? For which values of α is the method A-stable?

SECTION B

7 Write an essay on the connection between convergence, order and stability in the numerical solution of ordinary differential equations.

8 Describe in detail the fast Fourier transform and discuss its application to fast solution of the Poisson equation in a square.

9 Describe the Engquist-Osher method for a single, one dimensional, hyperbolic nonlinear conservation law

$$u_t + \frac{\partial}{\partial x} f(u) = 0.$$

Prove that it is stable, provided that f is convex, differentiable and possesses a unique stagnation (sonic) point.