UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2001 9 to 11

PAPER 55

DYNAMICAL SYSTEMS AND THERMODYNAMIC FORMALISM

*Attempt **TWO** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Show that a transformation

$$\begin{cases} x_{n+1} = \{\frac{1}{x_n}\}, & x_n \in [0,1] \\ y_{n+1} = \frac{1}{[\frac{1}{x_n}]+y_n}, & y_n \in [0,1] \end{cases}$$

of a unit square into itself has an absolute continuous invariant measure with a density

$$P(x,y) = \frac{1}{\log 2} \frac{1}{(1+xy)^2} \, .$$

Using the formula above find a density of absolute continuous invariant measure for the Gauss map

$$Tx = \{\frac{1}{x}\} \, .$$

(Notations $\{\cdot\}$, $[\cdot]$ above stand for fractional and integer part of a number.)

(ii) Let T_A be an automorphism of the $n-$dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$:

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad (\text{mod } 1) \, , \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in T^n \, ,$$

where $A \in SL(n, \mathbb{Z})$. Prove that T_A is mixing if and only if there are no roots of unity among the eigenvalues of the matrix A.

2

Let $U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots)$ be a potential such that for any two sequences which coincide on the first n places

$$|U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \epsilon^{(n+1)}, \ldots) - U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \tilde{\epsilon}^{(n+1)}, \ldots)| \leqslant C\lambda^{-n} ,$$

where $C > 0, \lambda > 1$ and spin variables $\epsilon_i \in K = \{1, 2, \ldots, k\}$. For every integer $s < t$ and every sequence $(\epsilon_i, i < s)$ define a probability distribution

$$p_{s,t}(\epsilon_s, \ldots, \epsilon_t | \epsilon_i, i < s) = \frac{1}{Z_{s,t}(\epsilon_i, i < s)} \exp(\sum_{j=s}^{t} U(\epsilon_j, \epsilon_{j-1}, \ldots)) ,$$

where $Z_{s,t}(\epsilon_i, i < s)$ is a normalization factor (partition function). A probability measure μ on $K^{\mathbb{Z}} = \{(\epsilon_i, i \in \mathbb{Z}), \epsilon_i \in K\}$ is called a Gibbs measure with a potential U if for all integer $s < t$ and for $\mu-$almost all $(\epsilon_i, i < s)$ the conditional probability under condition of fixed $(\epsilon_i, i < s)$ is equal to $p_{s,t}$:

$$\mu(\epsilon_s, \ldots, \epsilon_t | \epsilon_i, i < s) = p_{s,t}(\epsilon_s, \ldots, \epsilon_t | \epsilon_i, i < s) .$$

Prove uniqueness of a Gibbs measure μ.

(You may use Dobrushin's estimates for variational distances between probability distributions.)

3

Let T be an expanding mapping of the unit interval $[0,1]$, Denote by $\mathcal{E}$ the mapping corresponding to symbolic representation:

$$\mathcal{E} : [0,1] \to \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots)\} .$$

For every cylinder set $C_{\epsilon_1,\ldots,\epsilon_n}$ denote

$$\nu(\epsilon_1, \ldots, \epsilon_n) = \nu(C_{\epsilon_1,\ldots,\epsilon_n}) \, , \; l(\epsilon_1, \ldots, \epsilon_n) = l(\Delta^{(n)}_{\epsilon_1,\ldots,\epsilon_n}) \, ,$$

where ν is the Gibbs measure on the space of one-sided sequences $\{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots)\}$, l is the Lebesgue measure on $[0,1]$ and $\Delta^{(n)}_{\epsilon_1,\ldots,\epsilon_n}$ is an element of the partition of the $n-$th level corresponding to $(\epsilon_1, \ldots, \epsilon_n)$. Let

$$g_n(\epsilon_1, \ldots, \epsilon_n) = \log \frac{\nu(\epsilon_1, \ldots, \epsilon_n)}{l(\epsilon_1, \ldots, \epsilon_n)} \, .$$

(i) Show that there exists a function $g(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots)$ such that for all $\epsilon_1, \epsilon_2, \ldots$

$$|g(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) - g_n(\epsilon_1, \ldots, \epsilon_n)| \leqslant const \lambda^{-n} \, ,$$

where $\lambda = \min_{x \in [0,1]} |T^{'}(x)| > 1$.

(ii) Prove that $p(x) = \exp g(\mathcal{E}(x))$ is the density of an absolutely continuous invariant measure for T.

(You may use any results from the lectures on properties of the thermodynamic potentials for expanding maps without proof.)

4 (i) Let $g(x)$ be the Feigenbaum period-doubling map:

$$g(x) = \frac{1}{\beta} g(g(\beta x)) \,, g(0) = 1 \,, \beta = g(1) = -0.3995\ldots \,.$$

Using the period-doubling equation above find

$$\frac{dg}{dx}(1) \,.$$

(ii) Let $U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots)$ be the Feigenbaum potential. Show that there exists a limit

$$f(\gamma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\{\epsilon_i=0,1;\, 2\leqslant i\leqslant n\}} \exp\left(\gamma \sum_{j=1}^{n} U(\epsilon_j, \epsilon_{j-1}, \ldots, \epsilon_2, 1, 0, \ldots, 0, \ldots)\right) \,.$$

Prove that the function $f(\gamma)$ is continuous, monotone decreasing and convex. Also prove that there exists a unique γ_* such that $f(\gamma_*) = 0$.

(iii) Let $\Delta_i^{(n)}$, $1 \leqslant i \leqslant 2^n$ be the period-doubling partition of the $n-$th level. Prove that

$$f(\gamma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i=0}^{2^{n-1}} |\Delta_i^{(n)}|^{\gamma} \,.$$

Hence, show that

$$Dim_H(F) \leqslant \gamma_* \,,$$

where $Dim_H(F)$ is the Hausdorff dimension of the Feigenbaum attractor F.

(You may use an estimate

$$const \leqslant \frac{|\Delta_i^{(n)}|}{\exp\left(\sum_{j=1}^{n} U(\epsilon_j, \epsilon_{j-1}, \ldots, \epsilon_2, 1, 0, \ldots, 0, \ldots)\right)} \leqslant Const \,,$$

which holds for all odd $i = 1 + \sum_{i=2}^{n} \epsilon_i 2^{i-1}$.)