UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2001 9 to 12

PAPER 54

SYMMETRIES AND PATTERNS

*Attempt **THREE** questions. The questions are of equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the problem of steady-state bifurcation on a square lattice, with the usual rotation, reflection and translation symmetries. Suppose that the wavenumber vectors that generate the space of solutions are $\mathbf{k}_1 = (1,0)$ and $\mathbf{k}_2 = (0,1)$ but that the critical wavenumber for the bifurcation is $\sqrt{5}$, so that the eigenfunctions at onset are

$$A_1 e^{i(2x+y)} + A_2 e^{i(2x-y)} + A_3 e^{i(x+2y)} + A_4 e^{i(x-2y)} + c.c.$$

We define a *regular* nonlinear term in the A_j equation as a term of the form $A_j F(|A_1|^2, |A_2|^2, |A_3|^2, |A_4|^2)$ where F is a polynomial. It is given that there are no even-order terms in the normal form.

(a) By writing $A_j = R_j e^{i\theta_j}$, $j = 1, \ldots, 4$, or otherwise, show that because of the translation symmetries the centre manifold equation can be reduced to four real equations for the R_j, together with two other equations for the variables

$$\begin{aligned} \chi_1 &= 2\theta_1 - 2\theta_2 - \theta_3 + \theta_4 \\ \chi_2 &= \theta_1 + \theta_2 - 2\theta_3 - 2\theta_4 \ , \end{aligned}$$

and deduce that all the nonlinear terms at cubic order are regular. Use the symmetries of the lattice to write down the most general form of the normal form up to cubic order.

(b) Show that the only fifth order terms in the A_1 equation that are not regular are proportional to $A_2^* A_3^2 A_4^2$ and $A_1^* A_2^2 A_3 A_4^*$, where the asterisk denotes complex conjugate. Including these terms and their analogues for the other equations, but no other fifth order ones, find all possible distinct types of steady solution for which all the $|A_j|$ are equal.

2 The evolution equation for the vertically averaged temperature field $T(x,y,t)$ due to convection between poorly conducting boundaries takes the form

$$\frac{\partial T}{\partial t} = -\mu \nabla^2 T - \nabla^4 T + \nabla \cdot \left(|\nabla T|^2 \, \nabla T \right) , \qquad (*)$$

where $\nabla \equiv (\partial_x, \partial_y)$. Solutions are to be sought to this equation that are periodic on a square lattice of period 2π.

(a) Show that there is a steady-state bifurcation from the trivial solution at $\mu = 1$, with a four-dimensional centre manifold so that

$$T(x,y,t) = \epsilon(Ae^{ix} + Be^{iy}) \ + c.c. \ + h.o.t.$$

where ϵ is a small parameter. Now suppose that $\mu - 1 = \epsilon^2 \nu$; using perturbation theory, show that the normal form equations are

$$\begin{aligned} \dot{A} &= \nu A - \alpha |A|^2 A - \beta |B|^2 A \\ \dot{B} &= \nu B - \alpha |B|^2 B - \beta |A|^2 B \end{aligned}$$

where the constants α, β are to be found. Deduce that squares ($|A| = |B|$) are the stable planform near onset.

[*Any standard results derived in lectures concerning the stability of solutions of these equations may be quoted without proof*]

(b) Consider the functional

$$\mathcal{V}[T] \equiv \langle \frac{1}{2}\mu |\nabla T|^2 - \frac{1}{2}(\nabla^2 T)^2 - \frac{1}{4}|\nabla T|^4 \rangle \ ,$$

where $\langle \cdot \rangle$ denotes an average over a period of the lattice. Show that

$$\frac{d}{dt}\mathcal{V}[T] = \left\langle \left(\frac{dT}{dt} \right)^2 \right\rangle \ ,$$

and that $\mathcal{V}[T] \leqslant \mu^2/4$ ($\mu \geqslant 0$). What can you deduce about long-time solutions of (*)?

Consider now a spatially-periodic 'roll' solution of (*) when $\mu > 0$ of the form $T(x,y,t) = T_0(x)$ (You do not need to find this solution explicitly). Show that $\left\langle \mu (T_0')^2 - (T_0'')^2 - (T_0')^4 \right\rangle = 0$, and thus that for $\delta \ll 1$

$$\mathcal{V}[T_0(x) + \delta T_0(y)] - \mathcal{V}[T_0(x)] = \frac{\delta^2}{2}(\langle T_0'^4 \rangle - \langle T_0'^2 \rangle^2) + o(\delta^2).$$

[*You may assume that terms of $O(\delta)$ on the r.h.s. of this equation vanish*]

Deduce that $T = T_0(x)$ is unstable for all values of $\mu > 0$.

3 Consider a multiple Hopf bifurcation in one space dimension in the finite domain $-L < x < L$. The bifurcation structure depends on a parameter μ and on L. Suppose that at $\mu = \mu_c$, $L = L_c$ there are two marginal modes $Ae^{i\omega_1 t} f_1(x) + c.c.$, $Be^{i\omega_2 t} f_2(x) + c.c.$, where f_1 is even and f_2 is odd in x.

Show that the normal form for the dynamics on the extended (4-dimensional) centre manifold may be written (for (μ, L) close to (μ_c, L_c)),

$$\dot{A} = \nu_1 A - \alpha_1(|A|^2 + |B|^2)A - \beta_1 |B|^2 A$$
$$\dot{B} = \nu_2 B - \alpha_2(|A|^2 + |B|^2)B - \beta_2 |A|^2 B \ ,$$

where ν_i, α_i, β_i are complex constants.

Now suppose that $\omega_1 = \omega_2$ at onset. Deduce that the equations should be augmented by the extra terms

$$\dot{A} = \ldots - \gamma_1 B^2 A^*$$
$$\dot{B} = \ldots - \gamma_2 A^2 B^*$$

where the γ_i are complex constants, and the asterisk denotes complex conjugate.

Now consider the special case $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \beta$, with $\mathrm{Re}(\alpha) = \alpha_R > 0$, $\mathrm{Re}(\beta) = \beta_R < 0$, and write $\nu_2 = \nu_1 - d$, where $\mathrm{Re}(d) = d_R > 0$.

Investigate the linear stability of the periodic solution $A = Re^{i\Omega t}$, $B = 0$; $i\Omega = \nu_1 - \alpha R^2$ to perturbations in B. Show that the stability problem may be reduced to one with constant coefficients by writing $B = C(t)e^{i\Omega t}$. Give for each case conditions on β and d under which the reduced equations have

(a) a steady-state bifurcation at $R^2 = -\dfrac{|d|^2}{2\mathrm{Re}(\beta d^*)}$, and

(b) a Hopf bifurcation at $R^2 = -\dfrac{d_R}{\beta_R}$.

Without detailed calculation, give a description, in terms of the original variables, of the new solution branches that emerge from these bifurcations.

4 Write an essay about long-wavelength instabilities of periodic patterns. Your essay should include a treatment of the Eckhaus, zigzag and Benjamin-Feir instabilities. Discuss both instabilities near onset and the general problem for nonlinear patterns.