UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 12 June 2001 9 to 11

PAPER 50

ACOUSTICS AND STABILITY

*Attempt no more than **THREE** questions.*

Little credit will be given for fragments.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A thin elastic membrane of mass m per unit length is stretched along the x-axis under tension T. In $y > 0$ there is a quiescent fluid of density ρ_0 and wave-speed c_0, whereas in $y < 0$ there is a vacuum. Line source forcing of magnitude F and frequency ω is applied to the membrane at the origin, resulting in small oscillations such that the membrane's displacement $\eta(x, t)$ satisfies

$$m\frac{\partial^2\eta}{\partial t^2} - T\frac{\partial^2\eta}{\partial x^2} = F\delta(x)e^{-i\omega t} - p(x, 0, t),$$

where $p(x, y, t)$ is the perturbation pressure in the fluid.

Show that

$$\eta = \frac{F}{2\pi}\int_C \frac{\gamma e^{ikx}\, e^{-iwt}}{(Tk^2 - m\omega^2)\gamma - \rho_0\omega^2}\, dk$$

where $\gamma^2 = k^2 - k_0^2$ and $k_0 = \omega/c_0$, describing carefully the definitions of both γ and the inversion contour C. Obtain a similar expression for the fluid potential $\phi(x, y, t)$ and hence find the directivity of the far field radiation in the fluid.

[*You may use without proof the fact that*

$$\int f(k)e^{ikx-\gamma y}\, dk \sim \sqrt{2\pi k_0/r}\, f(k_0\cos\theta)\sin\theta\, e^{ik_0 r - i\pi/4}$$

as $r \to \infty$*, where* $x = r\cos\theta$*,* $y = r\sin\theta$ *and where the integration is along the steepest descent contour.*]

Show that, for a given real value of ω, the dispersion relation

$$D(k, \omega) = (Tk^2 - m\omega^2)\gamma - \rho_0\omega^2$$

always has two zeros on the real k-axis. To what do these zeros correspond? Are they always relevant?

2

Obtain Rayleigh's stability equation for an incompressible inviscid fluid in $-\infty < z < \infty$ with a basic flow $U(z)\mathbf{i}$ (where $\mathbf{i}$ denotes the unit vector parallel to the x-axis). State and prove Rayleigh's inflexion-point theorem. State without proof Fjørtoft's theorem.

Consider the basic flow $U(z) = \tanh z$. Show that the possibility that this flow is unstable is consistent with Fjørtoft's theorem. Approximate the basic flow using three straight segments:

$$U_{\text{approx}}(z) = \begin{cases} 1 & z > 1, \\ z & -1 < z < 1, \\ -1 & z < -1. \end{cases}$$

Find the dispersion relation corresponding to U_{approx} and hence or otherwise show explicitly that in this model the flow is unstable.

3 Attempt **EITHER** (a) **OR** (b).

(a) What is meant by a *singular perturbation problem*?

Functions $x(t)$ and $y(t)$ satisfy

$$\ddot{x} + 3\epsilon\dot{x}y + x = 2\dot{y}^2,$$
$$\dot{y} = \epsilon(1 + x - y),$$

subject to initial conditions $x(0) = 0$, $\dot{x}(0) = 1$, $y(0) = 3$, where $\epsilon \ll 1$ is a small parameter and a dot denotes differentiation with respect to t. By searching for a solution of the form

$$x = x_0(t) + \epsilon x_1(t) + \cdots,$$
$$y = y_0(t) + \epsilon y_1(t) + \cdots,$$

or otherwise, show that this perturbation problem is singular.

Show further that

$$x = f(\epsilon t)\sin t + O(\epsilon),$$
$$y = 1 + 2e^{-\epsilon t} + O(\epsilon)$$

is a uniformly valid solution for $\epsilon t \leqslant O(1)$, where f is a function to be determined.

(b) Consider the equation

$$\epsilon\frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + y = 0, \qquad (1)$$

with $y(0) = y(1) = 1$. Determine the solutions in the inner and outer regions up to and including terms of size $O(\epsilon)$, and find the corresponding additive composite solution.

Now consider the equation

$$\epsilon\frac{d^2y}{dx^2} - (1 + x)\frac{dy}{dx} + y = 0, \qquad (2)$$

with $y(0) = 1$, $y(1) = 1 + \epsilon$. Give a brief qualitative explanation of how the structure of the solution of equation (2) differs from that of the solution of equation (1). Determine the value of dy/dx at $x = 1$ up to and including only terms of size $O(1)$.

4 (a) Show how the Cole-Hopf transformation

$$q = 2\epsilon \frac{\partial}{\partial \theta} \ln \psi$$

can be used to transform Burgers' equation

$$\frac{\partial q}{\partial Z} - q\frac{\partial q}{\partial \theta} = \epsilon \frac{\partial^2 q}{\partial \theta^2}$$

into the Diffusion equation for ψ.

(b) Given that the general solution of the Diffusion equation is

$$\psi(\theta, Z) = \frac{1}{(4\pi\epsilon Z)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \psi(\theta', 0) \exp\left(-\frac{(\theta - \theta')^2}{4\epsilon Z}\right) d\theta',$$

show that the solution of Burgers' equation with initial data

$$q(\theta, 0) = \begin{cases} 0 & \theta < 0 \\ U & \theta > 0 \end{cases}$$

is

$$\frac{U}{1 + \alpha \exp\big(-U(\theta + \frac{1}{2}UZ)/2\epsilon\big)}$$

where

$$\alpha = \frac{\displaystyle\int_{\theta}^{\infty} \exp(-y^2/4\epsilon Z)\, dy}{\displaystyle\int_{-(\theta+UZ)}^{\infty} \exp(-y^2/4\epsilon Z)\, dy}.$$

Briefly describe what happens in the limit $\epsilon \to 0$, considering the cases $U > 0$ and $U < 0$ separately.

(c) The *Modified* Burgers' equation is

$$\frac{\partial q}{\partial Z} - q^2 \frac{\partial q}{\partial \theta} = \epsilon \frac{\partial^2 q}{\partial \theta^2}.$$

Consider travelling-wave solutions of the form $q = q(Z + c\theta)$ for c constant, such that $q \to 0$ as $Z \to -\infty$ and $q \to \beta$ as $Z \to \infty$, where β is a nonzero constant. Show that

$$\epsilon c^2 q' = q\left(1 - \frac{cq^2}{3}\right), \qquad (*)$$

where $'$ denotes differentiation with respect to argument. Hence determine c, and solve equation $(*)$ to find q.

5 (a) Starting from the equations of mass and momentum conservation, derive the equation

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \qquad (1)$$

where ρ is the density and the Lighthill tensor T_{ij} is to be defined. Given that the Green function for the wave equation is

$$\frac{\delta(t - |\mathbf{x}|/c_0)}{4\pi c_0^2 |\mathbf{x}|},$$

show that in the compact source limit, to be defined, the far-field density fluctuation is proportional to m^4, where m is the fluctuation Mach number.

(b) Explain what happens to equation (1) if additional mass sources of strength $M(\mathbf{x}, t)$ per unit time per unit volume are also present in the fluid, and find an expression for the corresponding far-field noise in the compact source limit.

Now consider a small bubble underwater which pulsates in a spherically symmetric manner with frequency ω. Without further detailed calculation, determine how the far-field acoustic density perturbation scales on ω.

(c) The simple wave equation describing acoustic propagation along a tube of cross-sectional area $A(Z)$ is

$$\frac{\partial q}{\partial Z} - q\frac{\partial q}{\partial \theta} + \frac{q}{2}\frac{\mathrm{d}}{\mathrm{d}Z}\ln(A) = 0. \qquad (2)$$

By making a transformation of the form $q(\theta, Z) = g(\zeta)Q(\theta, \zeta)$ with $\zeta = h(Z)$, where the functions g and h are to be determined, show that equation (2) becomes

$$\frac{\partial Q}{\partial \zeta} - Q\frac{\partial Q}{\partial \theta} = 0.$$

Spherically symmetric propagation in three dimensions is described by equation (2) with $A = Z^2$. For the general initial data $q(\theta, 0) = f(\theta)$, find an expression for the first positive value of Z for which a shock forms in equation (2).