UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 9 to 11

PAPER 48

LARGE-SCALE ATMOSPHERE–OCEAN DYNAMICS

*A distinction mark can be gained for complete and well-reasoned answers to **TWO** out of the three questions. The questions are of equal weight.*
Clarity and explicitness of reasoning will attract more credit than perfection of computational detail.

(x, y, z) denotes right-handed Cartesian coordinates and (u, v, w) the corresponding velocity components; t is time; the gravitational acceleration is $(0, 0, -g)$ where g is a positive constant.
The fluid is always incompressible, and always ideal in the sense that viscosity can be neglected, and likewise buoyancy diffusion if relevant. N denotes the buoyancy frequency of a stratified fluid.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define 'strictly geostrophic flow'. Prove the Taylor–Proudman theorem for such flow, concerning $\partial(p, \mathbf{u})/\partial z$ where z is the coordinate in the direction parallel to $\boldsymbol{\Omega} = (0, 0, \Omega)$, the angular velocity of the frame of reference, and where p and $\mathbf{u}$ are appropriate pressure and relative velocity fields.

State briefly (without justification) how the property of $\partial(p, \mathbf{u})/\partial z$ asserted by the Taylor–Proudman theorem must be modified when the flow is confined between rigid boundaries that are nearly but not quite perpendicular to $\boldsymbol{\Omega}$ and when the flow goes across geostrophic contours. State briefly (again without justification) how the z-component of vorticity must then behave, indicating how that behaviour is described by the vorticity equation.

Assume now that the two rigid boundaries are exactly perpendicular to $\boldsymbol{\Omega}$, and spaced apart by a constant distance h, except in a central circular area $x^2 + y^2 < a^2$ in which the spacing is reduced by a small constant amount ϵh. Fluid flows through the central area, the flow at large $r^2 = x^2 + y^2$ being uniform and unidirectional, with velocity $(U, 0, 0)$. Assuming that the z-component of vorticity behaves as before, find the relative vorticity field and show that the relative velocity field has streamfunction

$$\psi \ \propto \ \begin{cases} -\frac{1}{2}r^2 & (r < a), \\ a^2(\log a - \frac{1}{2} - \log r) & (r > a). \end{cases}$$

Find the constant of proportionality in terms of ϵ. Deduce the relative velocity field and find the value of ϵ/U such that closed streamlines are on the point of appearing in the relative velocity field.

2 Linearize the shallow-water momentum and mass-conservation equations about a state of relative rest and uniform depth h_{00}, in a frame of reference rotating with angular velocity $(0, 0, \frac{1}{2}f)$, where f is a positive constant. Deduce that the horizontal divergence $\delta = \nabla_H \cdot \mathbf{u} = \partial u/\partial x + \partial v/\partial y$ satisfies

$$\frac{\partial \delta}{\partial t} = fq - g\nabla_H^2 \zeta$$

where ζ is the surface elevation and q the relative vorticity. Deduce also that

$$\frac{\partial q}{\partial t} = -f\delta$$

and that

$$\frac{\partial}{\partial t}\left(\frac{q}{h_{00}} - \frac{f\zeta}{h_{00}^2}\right) = 0 \ .$$

Comment briefly on the significance of the quantity $(f + q)/(h_{00} + \zeta)$ for the *nonlinear* shallow-water equations, and on how is it related to the last equation above.

Show that small disturbances with the plane-wave structure $\exp(ikx + ily - i\omega t)$ satisfy the linearized equations provided that either $\omega = 0$ or

$$\omega^2 = f^2 + c_0^2(k^2 + l^2) \ ,$$

where c_0 is a constant to be defined. Sketch ω against $(k^2 + l^2)^{1/2}$ for this last relation and comment briefly on the way in which the phase and group velocities vary when ω takes values near and far from f. In the case of disturbances for which $\omega = 0$, how are the q and ζ fields related? In what qualitative way does the δ field differ from the q and ζ fields, when $\omega = 0$?

Briefly discuss what happens, according to the linearized equations, when the system is started from relative rest with surface elevation $\zeta = \epsilon h_{00}\exp(-x^2/L^2)$, where L is a given constant and $\epsilon \ll 1$.

3 Write an essay on quasi-geostrophic flow in a stratified, rotating system. Possible points are

- brief motivation;
- scaling assumptions and small parameters;
- scale-analytic arguments: 'thermal-wind relations' and representation of the velocity and buoyancy fields in terms of a geostrophic streamfunction ψ;
- simplification of the vorticity equation to describe the evolution of the motion with time;
- description of the evolution in terms of potential-vorticity advection and inversion;
- modification of Taylor-Proudman effects by stratification; Prandtl's ratio of scales;
- the possibility of, and implications of, a horizontal gradient of the Coriolis parameter f.