UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2001 9 to 11

PAPER 46

MIXING AND TRANSPORT

*Candidates may attempt **ALL** questions. All questions carry equal weight.*

A distinction mark will be awarded for complete, well-reasoned answers to two questions.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) A particle moves in one dimension under the influence of a stationary random velocity field $v(t)$, with ensemble average $\langle v(t)\rangle = 0$ at each t.

Let $X(t)$ be the position of the particle at time t. Derive an expression for $\langle (X(t) - X(0))^2\rangle$ in terms of the velocity autocorrelation function $\rho(t) = \langle v(0)v(t)\rangle / \langle v(0)v(0)\rangle$. Under what conditions is this expression consistent with diffusive behaviour as $t \to \infty$? Comment on the case $\rho(t) = (1 + |t|)^{-\alpha}$ for different values of the positive constant α.

(ii) An infinite channel $-\infty < x < \infty$, $-\frac{1}{2}L < y < \frac{1}{2}L$ contains an oscillating shear flow $u = U \cos\omega t \sin\frac{\pi y}{L}$ where ω, U and L are constants. Advection by this shear flow and diffusion (with constant diffusivity κ) act on a tracer with concentration $\chi(x, y, t)$. The channel walls are insulating, with $\partial\chi/\partial y = 0$ on $y = \pm\frac{1}{2}L$.

Assume that, at $t = 0$, $\chi = 1$ for $-\frac{1}{2}L < x < \frac{1}{2}L$, $-\frac{1}{2}L < y < \frac{1}{2}L$ and $\chi = 0$ elsewhere.

Consider the evolution of the tracer for $t > 0$ by constructing partial differential equations in y and t for the moments $[\chi]$, $[x\chi]$ and $[x^2\chi]$, where $[.] = \int_{-\infty}^{\infty}(.)dx$.

Show that at large times $[x^2\chi]$ increases linearly with time at rate $2Kt[\chi]$, where the constant K is to be determined as a function of U, ω, L and κ. Explain carefully any assumptions that you make.

Comment on the regime where $\omega L^2/\kappa$ is small.

2 A simple two-dimensional shear flow in the x-direction is defined by

$$(u, v) = (\Lambda y, 0)$$

where Λ is a constant.

Consider the action of this flow on an infinitesimal line element which is oriented in the direction $(\cos\theta, \sin\theta)$ at time $t = 0$. What is the orientation of the line element at a later time $t = T$ and by what factor has its length increased?

A 'renovating' shear flow has the form of the above shear flow in each interval $nT < t < (n+1)T$, where n is an integer, except that the direction of the flow changes randomly from one interval to the next. (The direction of the flow in the interval $nT < t < (n+1)T$ may be assumed to be independent of the flow at any previous times $t < nT$.

λ_N is the stretching factor for a given line element over the time $0 < t < NT$. What is the average stretching rate per unit time $\mu = \langle \log \lambda_N \rangle / NT$, expressed as Λ multiplied by a function of ΛT? The averaging operator $\langle . \rangle$ is to be taken over all initial orientations of the line element and all realisations of the subsequent flow. Justify carefully the steps in your calculation.

Describe the behaviour of μ as a function of ΛT. Show that there is a finite value of ΛT for which μ is a maximum. Explain your results including the behaviour for small and large ΛT, and the presence of a maximum, in qualitative terms.

What can you say about the probability distribution of λ_N for large N? Give as many quantitative details as you can about the distribution in the limit $\Lambda T \ll 1$.

[You may find the following results useful:

$$\int_0^{2\pi} \log(1 + \beta\cos\psi)\,d\psi = 2\pi \log\left(\frac{1 + \sqrt{1-\beta^2}}{2}\right) \text{ for } -1 < \beta < 1.$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^4\theta\,d\theta = \frac{3}{8}.$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^4\theta\cos^2\theta\,d\theta = \frac{1}{16}.$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^4\theta\cos^4\theta\,d\theta = \frac{3}{128}\ \Big].$$

3 Consider two-dimensional time-periodic flow defined by the streamfunction

$$\psi(\mathbf{x},t) = \psi_0(x,y) + \epsilon\psi_1(x,y,t) = -\sin x \sin y + \epsilon \cos x \cos y \sin \omega t,$$

where ϵ is a small constant.

The corresponding velocity field is

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x}) + \epsilon\mathbf{u}_1(\mathbf{x},t) = \left(-\frac{\partial\psi_0}{\partial y} - \epsilon\frac{\partial\psi_1}{\partial y}, \frac{\partial\psi_0}{\partial x} + \epsilon\frac{\partial\psi_1}{\partial x}\right).$$

(i) What does it mean to say that the unperturbed flow (with $\epsilon = 0$) is integrable? Sketch the streamlines of the flow with $\epsilon = 0$, identify the fixed points and indicate their stability. Also indicate the stable and unstable manifolds of the fixed points, where relevant. (You need consider only the region $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$, $-\frac{1}{2}\pi < y < \frac{3}{2}\pi$.)

(ii) Briefly describe the significance of intersections between stable and unstable manifolds in the perturbed flow (with $\epsilon > 0$).

(iii) The existence of intersections may be demonstrated, when ϵ is small, by the Melnikov perturbation technique. In this technique, the heteroclinic curve that joins the two hyperbolic points $\mathbf{x}_0^-$ and $\mathbf{x}_0^+$ of the unperturbed flow is parametrised by $\mathbf{x} = \mathbf{q}_0(s)$, $-\infty < s < \infty$, with $\mathbf{q}_0(0) = \mathbf{X}_0$, a specified reference point, and $d\mathbf{q}_0/ds = \mathbf{u}_0(\mathbf{q}_0(s))$. (This curve forms the unstable manifold of $\mathbf{x}_0^-$ and the stable manifold of $\mathbf{x}_0^+$.)

In the perturbed flow ($\epsilon > 0$) there are hyperbolic trajectories $\mathbf{x}^-(t)$ corresponding to $\mathbf{x}_0^-$ and $\mathbf{x}^+(t)$ corresponding to $\mathbf{x}_0^+$, where $\mathbf{x}^-(t)$ and $\mathbf{x}^+(t)$ are periodic functions of time.

Let $\mathbf{q}^s(t,t_0) = \mathbf{q}_0(t-t_0) + \epsilon\mathbf{q}_1^s(t,t_0) + O(\epsilon^2)$ for $t \geq t_0$, such that $\mathbf{q}^s(t,t_0) \to \mathbf{x}^+(t)$ as $t \to \infty$, be a trajectory in the stable manifold of $\mathbf{x}^+(t)$ and let $\mathbf{q}^u(t,t_0) = \mathbf{q}_0(t-t_0) + \epsilon\mathbf{q}_1^u(t,t_0) + O(\epsilon^2)$ for $t \leq t_0$, such that $\mathbf{q}^u(t,t_0) \to \mathbf{x}^-(t)$ as $t \to -\infty$, be a trajectory in the unstable manifold of $\mathbf{x}^-(t)$.

By considering the rate of change of $\nabla\psi_0(\mathbf{q}_0(t-t_0)).\mathbf{q}_1^s(t,t_0)$ and $\nabla\psi_0(\mathbf{q}_0(t-t_0)).\mathbf{q}_1^u(t,t_0)$ with respect to t show that the function $M(\mathbf{X}_0,t_0)$, defined by

$$M(\mathbf{X}_0,t_0) = \nabla\psi_0(\mathbf{q}_0(0)).(\mathbf{q}_1^u(t_0,t_0) - \mathbf{q}_1^s(t_0,t_0))$$

is given by the expression

$$M(\mathbf{X}_0,t_0) = \int_{-\infty}^{\infty} \nabla\psi_0(\mathbf{q}_0(\tau-t_0)).\mathbf{u}_1(\mathbf{q}_0(\tau-t_0),\tau)d\tau.$$

Explain why the existence of simple zeros of $M(\mathbf{X}_0,t_0)$ as $\mathbf{X}_0$ or t_0 varies implies the existence of transverse intersections between the stable and unstable manifolds of the unperturbed flow.

(iv) Consider the heteroclinic curve $y = 0$, $0 < x < \pi$ of the above flow with $\epsilon = 0$ and take $\mathbf{X}_0 = (X_0, 0)$.

Evaluate $M(\mathbf{X}_0,t_0)$. What does the form of $M(\mathbf{X}_0,t_0)$ imply about the transport properties of the perturbed flow?
[You may find it useful to note that $\int_{-\infty}^{\infty} \mathrm{sech}^2\, t \cos\alpha t.dt = \pi\alpha\, \mathrm{cosech}(\pi\alpha/2)$.]