UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2001 9 to 11

PAPER 45

THE FLUID DYNAMICS OF SWIMMING ORGANISMS

*Answer any **TWO** questions. The questions carry equal weight.*

Candidates may bring their notebooks into the examination.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Explain the principles of Lighthill's elongated body theory for a slender fish swimming at forward speed U, using small-amplitude undulations of its body. Show in particular that the lateral hydrodynamic force per unit length exerted on the water by the fish is

$$F_z(x,t) = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[m(x)\left(\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}\right)\right],$$

where $h(x,t)$ is the displacement of the fish centreplane, x is distance along the fish, t is time and $m(x)$ is a quantity to be defined. What is meant by the "recoil correction"?

(ii) A slender fish of length L is gliding parallel to itself at speed U, and then curves and straightens its spine once. The active bending can be represented by a centreplane displacement

$$h_0(x,t) = -\alpha(t)(x-\bar{x})^2, \quad 0 \leqslant x \leqslant L,$$

where $\bar{x}$ is the centre of mass of the fish, and $\alpha(t) > 0$ for $0 < t < T$, $\alpha(t) = 0$ otherwise. Find two simultaneous differential equations for the lateral displacement, $Z(t)$, of the fish's centre of mass and for the angle, $\theta(t)$, through which the fish rotates. [The coefficients of the equations will involve the fish's body mass, M_b, its moment of inertia about a vertical axis through the centre of mass, I_b, and the integrals

$$M_j = \int_0^L m(x)\,(x-\bar{x})^j\,dx, \quad j = 0,1,2,3.]$$

Integrate the equations once to obtain first order coupled equations for $\theta(t)$ and for $\gamma(t) = Z(t) + U\int_0^t \theta dt$.

Show that, after the manoeuvre, the fish will still be gliding parallel to itself and that it will have turned through an angle given by

$$\theta = 2U\int_0^T \alpha(t)dt.$$

Discuss the principal limitation(s) of the above model.

2 A dead, rigid spermatozoon sediments in a horizontal shear flow. The sperm head is a sphere of radius a; the tail is a cylinder of length L and radius b ($b \ll a \ll L$). The whole spermatozoon has a uniform density and its mass exceeds that of the water it displaces by m'; the centre of mass G is a distance h behind the centre C of the sphere. The fluid far away has horizontal velocity $\mathbf{U}^{\infty} = \alpha z \hat{\mathbf{e}}_1$, where α is a constant, $\hat{\mathbf{e}}_1$ is a horizontal unit vector, and z is measured vertically upwards. C is taken to be instantaneously at $z = 0$; the axis of the sperm tail lies in the vertical plane containing $\hat{\mathbf{e}}_1$ and makes an angle θ with the vertical. We seek to calculate the velocity $\mathbf{U}$ of C and the angular velocity Ω of the spermatozoon, using resistive force theory with force coefficients K_N (normal) and γK_N (tangential), where $K_N = 4\pi\mu/\sigma$ and $\sigma = O[\ln(L/b)] = O(L/a)$. The drag on the sphere is $-6\pi\mu a\mathbf{U}$, and the torque on a rotating sphere is $-\frac{8}{3}\pi\mu a^3$ multiplied by the angular velocity relative to the fluid.

Show that the velocity of a point on the tail at distance s from its point of attachment to the sphere, relative to the fluid "far away" (i.e. a distance $\gg b$ from the point), has normal component

$$\mathbf{U} \cdot \hat{\mathbf{j}} + \left(\Omega + \alpha \cos^2\theta\right)(a + s),$$

where $\hat{\mathbf{j}}$ is a unit vector in the normal direction (see diagram), and write down the tangential component.

From the force balance in the $\hat{\mathbf{j}}$-direction and the torque balance about C, derive the following approximate expression for Ω, explaining all approximations made:

$$\Omega \approx 2\beta \sin\theta - \alpha \cos^2\theta \quad \text{where} \quad \beta = \frac{m'g}{\left(\frac{K_N L^2}{3} + 8\pi\mu a L\right)}. \tag{1}$$

Deduce from (1) that there are two equilibrium orientations, only one of which is stable. In the stable orientation, is the head lower or higher than the far end of the tail?

Calculate the corresponding expressions for $\mathbf{U} \cdot \hat{\mathbf{i}}$ and $\mathbf{U} \cdot \hat{\mathbf{j}}$ ($\hat{\mathbf{i}}$ is parallel to the sperm tail). If the organism is sedimenting in its equilibrium orientation, does increasing the shear rate α increase or decrease the vertical velocity?

3 A suspension of spherical, bottom-heavy, micro-organisms (cells) occupies a chamber of depth h. The swimming direction of a cell, represented by the unit vector $\hat{\mathbf{p}}$, is given, deterministically, by a balance between viscous and gravitational torques; randomness in the cells' swimming is modelled by means of an isotropic cell diffusivity D. Explain every term in the following conservation equation:

$$\frac{\partial \bar{n}}{\partial \bar{t}} = -\bar{\nabla} \cdot \left[\bar{n}(\bar{\mathbf{u}} + V_c \hat{\mathbf{p}}) - D\bar{\nabla}\bar{n} \right],$$

where $\bar{n}(\bar{\mathbf{x}}, \bar{t})$ is the number of cells per unit volume and an over-bar represents a dimensional variable. Justify the neglect of an individual cell's settling velocity. Also write down and explain the equation relating $\hat{\mathbf{p}}$ to the local fluid vorticity, $\bar{\boldsymbol{\omega}} = \bar{\nabla}_\wedge \bar{\mathbf{u}}$. Given that the suspension as a whole behaves like a Newtonian fluid with constant viscosity, state the other equations that are needed for an analysis of the motion, explaining any additional notation.

Calculate the steady-state distribution of cells when the suspension as a whole is at rest, showing that

$$\bar{n} = \bar{n}_0(\bar{z}) = \bar{N}_0 e^{V_c \bar{z}/D},$$

where $\bar{z}$ is the vertical coordinate, measured upwards from the bottom of the chamber. What is the relationship of the constant $\bar{N}_0$ to the average cell concentration $\bar{N}_{00}$?

Analyse the linear stability of this steady state: non-dimensionalise the variables using $\bar{N}_0$ as the scale for $\hat{n}$, h as the scale for lengths, D/h for velocities and h^2/D for time, and assuming that perturbations to the steady state have order of magnitude ϵ and real horizontal wave number vector $(l, m, 0)$, so that, for example,

$$\begin{aligned} n(x,y,z,t) &= n_0(z) + \epsilon N(z) \exp[\sigma t + ilx + imy], \\ w(x,y,z,t) &= \qquad\quad\;\; \epsilon W(z) \exp[\sigma t + ilx + imy] \end{aligned}$$

etc. Here variables without the overbar are dimensionless, w is the vertical component of the dimensionless fluid velocity $\mathbf{u}$, and σ is the (possibly complex) growth rate. Show that, to first order in ϵ, the governing equations reduce to

$$\left(\frac{d^2}{dz^2} - \beta \frac{d}{dz} - k^2 - \sigma \right) N = -\beta e^{\beta z} \left[G \left(\frac{d^2 W}{dz^2} - k^2 W \right) - W \right] \tag{1}$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{\sigma}{Sc} \right) \left(\frac{d^2 W}{dz^2} - k^2 W \right) = -\frac{R}{\beta} k^2 N, \tag{2}$$

where $k^2 = l^2 + m^2$, $\beta = hV_c/D$, $G = BD/h^2$, $Sc = \nu/D$ and

$$R = \frac{\beta h^3 \upsilon \Delta\rho g \bar{N}_0}{\rho \nu D}$$

(where B = gyrotactic reorientation time, υ = cell volume, $(\rho + \Delta\rho)$ = cell density, ρ = water density, g = gravitational acceleration, ν = fluid kinematic viscosity). Write down the boundary conditions to be satisfied by N and W, given that there are rigid boundaries at $z = 0, 1$.

Now investigate long wavelength disturbances in very shallow chambers, for which $\beta \ll 1$. Assume that the scalings as $\beta \to 0$ are $k = \beta\alpha$, $\sigma = \beta^2 \sigma_2$, $G = \beta^{-1} G_{-1}$, $R = \sum_{j=0}^{\infty} \beta^j R_j$, $N = \sum_{j=0}^{\infty} \beta^j N_j(z)$, $W = \sum_{j=1}^{\infty} \beta^j W_j(z)$ and expand equations (1) and

(2) in powers of β. Find N_0 and W_1 and show, by integrating the $O(\beta^2)$ term from (1) between $z = 0$ and $z = 1$, that

$$\alpha^2 + \sigma_2 = -\int_0^1 W_1 dz.$$

Deduce that

$$\sigma_2 = \alpha^2 \left(\frac{R_0}{720} - 1 \right).$$

What does this theory tell you about the onset of bioconvection in shallow chambers?