UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 1.30 to 4.30

PAPER 44

SLOW VISCOUS FLOW

*Candidates may attempt **ANY NUMBER** of questions, but substantially complete answers will be viewed more favourably than fragments.*

*Full marks can be gained by complete answers to **THREE** questions.*

The questions are of equal weight.

Do not start reading the questions printed on the subsequent pages until instructed to do so by the invigilator.

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1 The annular region between two concentric rigid spheres of radii a and b (with $b > a$) is filled with fluid of viscosity μ. The outer sphere is held stationary, while the inner sphere is made to rotate with angular velocity $\mathbf{\Omega}$.

Assuming that inertia is negligible, use the Papkovich-Neuber representation

$$\mathbf{u} = \nabla(\mathbf{x} \cdot \mathbf{\Phi}) - 2\mathbf{\Phi} \, , \qquad p = 2\mu\nabla \cdot \mathbf{\Phi}$$

to determine the fluid velocity. [*Hint: Choose* $\mathbf{\Phi}$ *as a linear combination of two simple vector harmonic functions.*]

Show explicitly that your solution gives $p = 0$. Explain how this result could also have been obtained without detailed calculation from simple properties of Stokes flow.

Calculate the stress field in the fluid. Deduce the couple $\mathbf{G}$ that must be applied to the inner sphere to maintain the motion.

What is the condition that inertia is indeed negligible for the case $b \gg a$? Explain briefly why this is not the appropriate condition for the case $b - a \ll a$.

State the Minimum Dissipation Theorem for Stokes flow, making it clear which flows are compared by the theorem.

A number of force-free, couple-free rigid particles are added to the fluid between the spheres, but the concentric position and relative angular velocity $\mathbf{\Omega}$ of the inner sphere are maintained by application of the necessary force and couple to the inner sphere. Show that the component of the couple in the direction of $\mathbf{\Omega}$ is increased, being careful to explain each step of the argument.

2 A vertical planar sheet of very viscous fluid of density ρ and viscosity μ undergoes extension. With respect to Cartesian axes, the sheet occupies $-\frac{1}{2}h(z,t) \le x \le \frac{1}{2}h(z,t)$. There is no flow or variation in the y-direction, so that the velocity $\mathbf{u}(x,z,t) = (u,0,w)$. Gravity acts in the positive z-direction and surface tension is negligible. The sheet is surrounded on both sides by inviscid fluid of density ρ_a in which there is a hydrostatic pressure gradient $p_a(z) = p_0 + \rho_a g z$.

Assuming that $\partial h/\partial z \ll 1$, explain why w is approximately independent of x and derive equations for $u(x,z,t)$ and σ_{xx}. Deduce that $\sigma_{zz} = 4\mu(\partial w/\partial z) - p_a(z)$.

Draw a diagram to show all the forces acting on a fluid slice of length δz and varying width. Deduce that

$$\frac{4\mu}{h}\frac{\partial}{\partial z}\left(h\frac{\partial w}{\partial z}\right) + (\rho - \rho_a)g = 0 \ .$$

Use conservation of mass to obtain another relationship between $h(z,t)$ and $w(z,t)$.

Now let $\rho = \rho_a$ (or, equivalently, suppose that gravity is negligible). At $t = 0$ the sheet has length $2L_0$ and thickness $h_0(z)$. For $t > 0$ the sheet is stretched to length $2L(t)$ by pulling on the ends at $z = \pm L(t)$ with equal and opposite forces $\pm F(t)$ (per unit width in the y-direction). Let $z_0(z,t)$ denote the initial ($t = 0$) position of the fluid element which is at position z at time t. For example, $z_0(L,t) = L_0$. You may assume that $z_0(0,t) = 0$.

By considering the evolution of a fluid element, show that

$$h(z,t) = h_0(z_0) - \Delta(t) \qquad \text{where} \qquad \Delta(t) = \frac{1}{4\mu}\int_0^t F(t')\,dt'.$$

Use conservation of mass to express $\dfrac{\partial z_0}{\partial z}$ in terms of h and h_0. Deduce that

$$z = z_0 + \int_0^{z_0} \frac{\Delta(t)\,dz_0'}{h_0(z_0') - \Delta(t)} \ .$$

Find L as a function of Δ when $h_0(z) = H(1 + k^2 z^2)$, where H and k are constants. [*Note that* $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$.]

If F is constant show that L becomes infinite after a finite time t^*, and determine t^*.

Show also that $L \sim A(t^* - t)^\alpha$ as $t \to t^*$, and find the constants A and α for the two cases $k \neq 0$ and $k = 0$. Comment briefly on why the values of α differ between the cases and explain why A does not depend on L_0 for $k \neq 0$.

3 A thin film of viscous fluid of thickness $h(x,t)$ lies between a hot planar rigid boundary $z=0$ and a cold inviscid environment. The temperature $T(x,z,t)$ of the fluid satisfies

$$\frac{DT}{Dt} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) \quad \text{in } \ 0 < z < h(x,t) \ , \tag{1}$$

$$T = T_0 \text{ at } z=0 \qquad \text{and} \qquad -\kappa\frac{\partial T}{\partial z} = \alpha[T - (T_0 - \Delta T)] \text{ at } z = h(x,t) \ ,$$

corresponding to a fixed boundary temperature T_0 and conductive cooling to an environmental temperature $T_0 - \Delta T$. Here α and κ are constants.

Find the steady temperature distribution $T(z)$ when h is uniform and the fluid velocity $\mathbf{u}$ is zero. Show that if $\alpha h/\kappa \ll 1$ then

$$T(h) \approx T_0 - \Delta T\frac{\alpha h}{\kappa} \ . \tag{2}$$

Use scaling on the terms in (1) to show that (2) still holds when h varies on a lengthscale L and the x-component of $\mathbf{u}$ has typical magnitude U, provided that $\epsilon^2 \equiv (h/L)^2 \ll 1$ and $Pe \equiv Uh^2/(\kappa L) \ll 1$.

The fluid has a temperature-dependent surface tension $\gamma(T) = \gamma_0 + \gamma'(T - T_0)$, where $\gamma_0 > 0$ and $\gamma' < 0$ are constants. The other properties of the fluid are independent of temperature, and gravity is negligible. Assuming that the surface temperature is given by (1), use lubrication theory to show that

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\gamma_0}{3\mu}h^3\frac{\partial^3 h}{\partial x^3} - \frac{\gamma'\Delta T\alpha}{2\kappa\mu}h^2\frac{\partial h}{\partial x}\right) = 0 \ . \tag{3}$$

[*Justification of the approximations in lubrication theory is* **not** *required. You may assume that the surface curvature is approximately $\partial^2 h/\partial x^2$, but the use of γ_0 in the second term instead of $\gamma(T)$ should be justified by a scaling argument.*]

Give a brief *physical* explanation why the second and third terms respectively cause perturbations to a uniform film thickness to decay and to grow.

Equation (3) can be reduced to the dimensionless form

$$H_\tau + (H^2 H_X)_X + (H^3 H_{XXX})_X = 0 \tag{4}$$

by defining $H = h/\hat{h}$, $X = \epsilon x/\hat{h}$ and $\tau = t/\hat{t}$. Find the timescale $\hat{t}$ and aspect ratio ϵ.

Obtain and sketch the dispersion relationship $s(k)$ for small disturbances of the form $H = 1 + \delta\exp(s\tau + ikX)$ with $\delta \ll 1$. What is the most unstable wavenumber?

Show that steady solutions of (4) with zero net flux satisfy

$$\tfrac{1}{2}H_X^2 + V(H) = E,$$

where $V(H) = H(\ln H - 1)$ and E is a constant, provided that $\hat{h}$ has been chosen so that $H = 1$ when $H_{XX} = 0$.

4 Define the Darcy velocity **u** and the pore velocity **v** for flow in a random porous medium. State the relationship between these velocities and the porosity ϕ? State the condition of incompressible flow, when ϕ is independent of time but varies with position. What is the average velocity of a fluid interface within the medium (neglecting any effects of dispersion)?

An impermeable plane, inclined at angle θ to the horizontal, is overlain by a porous medium with porosity $\phi(z) = Pz^{\alpha-1}$ and permeability $k(z) = Kz^{\beta-1}$, where P, K, α and β are positive constants and z is the *perpendicular* distance to the impermeable plane. Let x and y be the corresponding down-slope and cross-slope coordinates.

Use a lubrication-like approximation to show that gravity-driven flow over the plane through the porous medium of a thin layer of fluid of slowly varying thickness $h(x, y, t)$ is described by the equation

$$\frac{\partial h^\alpha}{\partial T} + \frac{\partial h^\beta}{\partial X} = \frac{\partial}{\partial X}\left(h^\beta \frac{\partial h}{\partial X}\right) + \frac{\partial}{\partial Y}\left(h^\beta \frac{\partial h}{\partial Y}\right), \tag{1}$$

where $X = x \tan\theta$, $Y = y \tan\theta$ and $T = t(\alpha\rho g K \sin\theta \tan\theta)/(\beta\mu P)$. The fluid density and viscosity are ρ and μ, and the pressure is constant in the region above the fluid layer.

(i) Consider *steady* flow from a point source at the origin, which feeds a down-slope current with constant flux and finite cross-slope width $Y_N(X)$. Sufficiently far down-slope, $Y_N \ll X$.

By making suitable approximations and scaling estimates, show that $Y_N \propto X^\gamma$ and find γ. [*You do* **not** *need to calculate the detailed similarity solution.*]

(ii) Now consider *time-dependent* flow for the case $\alpha = \beta = 1$ (constant porosity and permeability). What is the form of (1) after the variable transformation $\tau = T$, $\xi = X - T$, $\eta = Y$?

Consider the release of a fixed volume V of fluid at $x = y = 0$ at $t = 0$. Show that

$$V = P \tan^2\theta \int h \, d\xi \, d\eta \, .$$

Derive the detailed axisymmetric similarity solution for h.

[*In plane-polar coordinates* $\nabla^2 f(r) = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial f}{\partial r}\right)$.]

[END OF PAPER]

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