UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2001 9 to 12

PAPER 43

ENVIRONMENTAL FLUID DYNAMICS

*Answer **TWO** questions from Section A and **ONE** from Section B.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 Consider the flow of a fluid of constant density ρ_1 along a horizontal channel of varying width $b(x)$. The ambient fluid has density ρ_0.

(a) State the Boussinesq assumption and the conditions required for shallow water flow. By considering fluxes of mass and momentum through a control volume, or otherwise, derive the single-layer shallow water equations for an inviscid fluid, and show that the system is hyperbolic. Determine the characteristics and the equations describing variations in the flow along the characteristics.

(b) A volume of fluid V is released in a channel of unit width. The channel is closed at $x = 0$. Sketch the flow that develops, indicating the key features of the flow. Draw also an $x - t$ diagram to illustrate the propagation of information and any key transitions the flow undergoes.

(c) At late time, the gravity current will tend towards an exact solution of the shallow water equations where the depth and velocity may be written as $h(x,t) = h_f(t)H(x/L)$ and $u(x,t) = u_f(t)U(x/L)$, where $L = L(t)$ is the length of the current, $u_f = dL/dt$, $H(1) = 1$ and $U(1) = 1$. Show that $L \, dh_f/dt = -h_f \, dL/dt$ and state a suitable boundary condition for the front. Determine the complete solution. (Hint: rewrite the shallow water equations in terms of time t and the similarity variable $\eta = x/L$.)

2 Small amplitude perturbations to an inviscid, non-diffusive incompressible fluid with buoyancy frequency $N(z)$, at rest in the mean, satisfy

$$\nabla.\mathbf{u} = 0, \qquad \frac{\partial \sigma}{\partial t} = -wN^2 \qquad \text{and} \qquad \frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla P + \sigma\hat{\mathbf{z}},$$

where $\mathbf{u} = (u, v, w)$ is the velocity, $\hat{\mathbf{z}}$ the vertical unit vector (positive upwards), σ is the perturbation to the mean buoyancy $\overline{b} = -g(\overline{\rho}(z) - \rho_0)/\rho_0$, and P is the perturbation to the mean hydrostatic pressure, and g is the acceleration due to gravity.

(a) Derive the equation governing the amplitude of the vertical velocity perturbation of a 2D normal mode internal gravity wave ($\sim e^{i(kx-\omega t)}$) in this fluid.

(b) Express the vertical displacement of a fluid parcel η, and the modified pressure P, in terms of w and dw/dz for this mode. Hence derive two continuity conditions for w.

A plane internal gravity wave with vertical velocity perturbation $w(\mathbf{x}, t) = w_0 e^{i(kx - m_0 z - \omega t)}$ is incident from far below in the stratification

$$\overline{b}(z) = \begin{cases} b_1, & \text{for } z > 0; \\ b_1 - \Delta b + N_0^2 z, & \text{for } z < 0, \end{cases}$$

where m_0, k, ω, b_1, Δb and $N_0 > \omega$ are positive constants.

(c) Express m_0 in terms of k, N_0 and ω. At what value of the interface strength $S = \Delta b k/\omega^2$ is the largest disturbance observed in the upper layer? At this value of S, how does the forcing frequency compare with the frequency of free waves on an interface between two well-mixed layers, $\omega_{free} = \sqrt{\Delta b k/2}$?

(d) Instead, assume that the well-mixed layer absorbs all incident wave energy, 1/4 of which is converted to potential energy, by deepening of the mixed layer. Calculate the rate of descent of the interface at $z = -d(t)$, where $d(0) = 0$.

3 A turbulent plume with a 'top hat' profile of radius b, vertical velocity w and density ρ_p rises from a point source of buoyancy B_0 into ambient fluid of density ρ.

(a) Define g', the reduced gravity of the plume, and give expressions for the volume flux Q, momentum flux per unit mass M and buoyancy flux B. Describe the entrainment hypothesis and define the entrainment coefficient α. Assuming the ambient density is constant, determine the scaling for Q, M and B as a function of $B_0 > 0$ and the height z above the source.

(b) Explain why a plume in stratified ambient fluid with constant $N^2 = -(g/\rho_0)(d\rho/dz) > 0$ has a finite rise height H. Sketch this plume and determine the scaling for H.

(c) Consider the ambient stratification given by $N^2 = N_s^2(z/z_s)^\beta$, where $N_s^2 > 0$, $z_s > 0$ and β are constant. What is the physical significance of $\lambda = H/z_s$ (where H is the rise height for $\beta = 0$)? For nonzero β the plume can either rise to a height H' or it can continue to rise indefinitely, depending on λ and β. Using physical arguements, explain how H'/H should vary with λ and β for cases of a finite rise height.

(d) The nondimensional plume equations may be written as

$$\frac{d\tilde{Q}}{d\tilde{z}} = \tilde{M}^{1/2}, \qquad \frac{d\tilde{M}}{d\tilde{z}} = \tilde{B}\tilde{Q}/\tilde{M}, \qquad \frac{d\tilde{B}}{d\tilde{z}} = -\tilde{Q}(\lambda\tilde{z})^\beta$$

where the tildes ($\tilde{}$) indicate the dimensionless variables, and $\tilde{z} = z/H$. Show that these equations allow similarity solutions of the form $\tilde{Q} = c_\theta\tilde{z}^\theta$, $\tilde{M} = c_\mu\tilde{z}^\mu$, $\tilde{B} = c_\psi\tilde{z}^\psi$, and determine c_θ, c_μ, c_ψ, θ, μ and ψ. Determine the range of β for which $\tilde{B}$ remains finite if the source is located at $\tilde{z} = 1/\lambda$. Comment on the form of the solution as $\tilde{z} \to \infty$ and explain in physical terms why the plume can rise unbounded to infinity.

Section B

4 A sheared semi-infinite turbulent fluid layer in $z > 0$ is forced by an applied shear stress $\tau_0 = \rho_0 U_*^2$ in the x-direction, where U_* is the friction velocity. A buoyancy flux per unit area of B_0 is applied at $z = 0$. The ensemble mean buoyancy $\overline{b}$ and velocity $\overline{\mathbf{u}}$ satisfy

$$\nabla.\overline{\mathbf{u}} = 0, \qquad \left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}.\nabla\right)\overline{b} = -\nabla.\left(-\kappa\nabla\overline{b} + \overline{\mathbf{u}'b'}\right)$$

and $$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}.\nabla\right)\overline{u}_i = -\frac{1}{\rho_0}\frac{\partial\overline{p}}{\partial x_i} + \overline{b}\,\delta_{i3} - \nabla.\left(-\nu\nabla\overline{u}_i + \overline{\mathbf{u}'u_i'}\right),$$

where $b' = b - \overline{b}$, $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$, and the molecular viscosity ν and diffusivity κ are constant.

(a) Simplify these equations for a horizontally homogeneous and stationary flow, for which $\overline{w} = 0$ at $z = 0$. Explain briefly why the turbulent correlations typically lead to down-gradient fluxes, and define the vertical turbulent viscosity K_M and diffusivity K_B. What further simplifications can be made if the Reynolds and Peclet numbers are large?

(b) Assume that K_M and K_B depend only on the parameters U_*, z and the Monin-Obukhov length $l_m \sim -U_*^3/B_0$. Show that the mean velocity and buoyancy profiles take a 'log-linear' form for $z/l_m \ll 1$. By calculating the Richardson number, or otherwise, give a physical interpretation of the limit $z/l_m \ll 1$.

(c) The buoyancy flux B_0 is provided by a well-mixed hot salty reservoir fed from a hydrothermal vent supplying fluid at flow rate Q, salinity S_i and temperature T_i. The reservoir is maintained at a steady state by flow over a weir, and has volume V and plan area A. The salinity and temperature just above the reservoir are S_∞ and T_∞, both constant. The stability of the double diffusive interface above the reservoir is measured by $R_\rho = \beta\Delta S/\alpha\Delta T$ where β and α are the constant salinity contraction and thermal expansion coefficients, respectively, and ΔS and ΔT are the salinity and temperature jumps across the interface. What is the minimum value of R_ρ? Why? What happens to the double diffusive convection as $R_\rho \to \infty$?

Question 4 continues overleaf.

(d) The total buoyancy flux per unit area across the double diffusive interface is $B_0 = g(\alpha F_T - \beta F_S)$. The double diffusive fluxes of salt and heat, F_S and F_T, are paramaterised using

$$\frac{\beta F_S}{\alpha F_T} = R_F \qquad \text{and} \qquad \alpha F_T = b(\alpha \Delta T)^{4/3}/R_\rho^2,$$

where b is a dimensional constant and R_F is a dimensionless constant. Introduce the dimensionless variables $\eta = (S_i - S)/(S_i - S_\infty)$ and $\theta = (T_i - T)/(T_i - T_\infty)$, to show that the dimensionless steady state salinity η_s and temperature θ_s in the reservoir can be written as

$$\eta_s = \theta_s \frac{R_f}{R_\rho^o} \qquad \text{and} \qquad \theta_s = C(1-\theta_s)^{10/3}/(1-\eta_s)^2,$$

where $R_\rho^o = \beta(S_i - S_\infty)/\alpha(T_i - T_\infty)$. What is the value of C?

(e) At time $t = t_0$ the hydrothermal vent closes and flow out of the reservoir immediately ceases. Assuming that the interface above the reservoir remains at the level of the weir, examine the transient evolution of η and θ to determine $\eta = \eta(\theta)$ and $t = t(\theta)$.

5 A room of height H and floor area S is connected to the outside by two vents each of area A with discharge coefficients C_d. One vent is located at ceiling level and the other at floor level, and outside the air has density ρ_0.

(a) The vents are opened at $t = 0$ allowing the warm air of density $\rho_0 - \Delta\rho$ (where $\Delta\rho \ll \rho_0$) within the room to escape. Determine the time evolution of the depth h of the layer of warm air within the room, stating all necessary assumptions.

(b) A localised heat source with buoyancy flux B_0 is turned on at floor level in the centre of the room and a steady state develops. Sketch this flow and determine an implicit expressions for the depth h and density $\rho_0 - \Delta\rho$ of the warm air layer. The reduced gravity in the plume above the heat source satisfies

$$g'_p = \gamma B_0^{2/3} z^{-5/3},$$

where z is the height above the source and γ is a constant.

(c) On another day, a doorway to the outside of height D and width W_0 is opened instead of the vents. The resulting flow of warm air out through the doorway may be modelled as a single-layer hydraulic flow over a broad-crested weir. State what is meant by the terms 'supercritical', 'subcritical' and 'hydraulic control'. Why does the single-layer hydraulic model break down if the warm air layer extends too far below the top of the doorway? Outside the room the flow accelerates upward away from the doorway and can no longer be described by hydraulic theory. Why does this not matter?

(d) Derive the specific energy function E for an inverted channel of varying width $W(x)$ and top elevation $H(x)$ containing a layer of depth $h(x)$ and density $\rho_0 - \Delta\rho$, stating any assumptions made. Determine the relationship between $\partial E/\partial h$ and the local Froude number F. For an inverted channel of uniform width containing a single weir, use a series of sketches to describe the different hydraulic flows that may exist and their relationship with E.

(e) Determine the relationship between the thickness and density of the warm air layer within the room and the flux of warm air through the doorway. Hence or otherwise determine an implicit expression for the steady thickness and density of the warm air layer when the heat source used in (b) is turned on. What role is played by the air above the doorway if $D < H$?

(f) Suppose a conservatory is added outside the room so that the warm air layer flows into the conservatory and then through a second doorway to the outside. This doorway is the same height but a different width to the first doorway. Outline using sketches the possible flow configurations that exist depending on the relative width of the doorways. You need not give analytical expressions, but you should describe any special features of the flow.