UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2001 1.30 to 3.30

PAPER 42

ACCRETION DISCS

Answer any ***TWO*** *questions. The questions are of equal weight.*

Candidates are reminded of the following expressions in cylindrical polar coordinates (r, ϕ, z)*, where* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are vectors, and* $\boldsymbol{C}$ *is a symmetric, second-rank tensor:*

$$
\begin{aligned}
\nabla \cdot \boldsymbol{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}, \\
\boldsymbol{A} \cdot \nabla \boldsymbol{B} &= \boldsymbol{e}_r\left(\boldsymbol{A}\cdot\nabla B_r - \frac{A_\phi B_\phi}{r}\right) + \boldsymbol{e}_\phi\left(\boldsymbol{A}\cdot\nabla B_\phi + \frac{A_\phi B_r}{r}\right) + \boldsymbol{e}_z\left(\boldsymbol{A}\cdot\nabla B_z\right), \\
\nabla \cdot C &= \boldsymbol{e}_r\left[\frac{1}{r}\frac{\partial}{\partial r}(rC_{rr}) + \frac{1}{r}\frac{\partial C_{r\phi}}{\partial \phi} + \frac{\partial C_{rz}}{\partial z} - \frac{C_{\phi\phi}}{r}\right] \\
&\quad + \boldsymbol{e}_\phi\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 C_{r\phi}\right) + \frac{1}{r}\frac{\partial C_{\phi\phi}}{\partial \phi} + \frac{\partial C_{\phi z}}{\partial z}\right] \\
&\quad + \boldsymbol{e}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rC_{rz}) + \frac{1}{r}\frac{\partial C_{\phi z}}{\partial \phi} + \frac{\partial C_{zz}}{\partial z}\right].
\end{aligned}
$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Starting from the equation of mass conservation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0,$$

and the equation of motion in the form

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\rho \nabla \Phi - \nabla p + \nabla \cdot \mathbf{T},$$

derive the diffusion equation for the surface density of a Keplerian accretion disc,

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}\bar{\nu}\Sigma)\right]. \qquad (1)$$

You should state clearly any assumptions you make, and define the mean effective kinematic viscosity $\bar{\nu}$ in terms of an appropriate integral of the stress tensor $\mathbf{T}$.

A Keplerian accretion disc has the viscosity law $\bar{\nu} = Ar$, where A is a constant. Show that equation (1) can be transformed into

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2} \qquad (2)$$

by an appropriate choice of variables. Obtain a similarity solution for a disc satisfying a zero-torque boundary condition at $r = 0$. Show that the mass of the disc decreases in time according to a power law.

Verify that

$$g = t^{-1/2} e^{-x^2/4t}$$

also satisfies equation (2), and discuss the physical interpretation of this solution.

2 An inviscid, perfectly conducting, incompressible fluid of uniform density ρ occupies the region $a < r < b$ between rigid cylindrical boundaries, where (r, ϕ, z) are cylindrical polar coordinates. The system is governed by the equations

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla\Pi + \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\boldsymbol{B},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u},$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

$$\nabla\cdot\boldsymbol{B} = 0.$$

Explain the meaning of Π in the first equation, and write down an appropriate set of boundary conditions for this system.

Show that there are steady solutions in which the velocity and magnetic field have only azimuthal components that depend only on radius. Obtain the most general solution for $u_\phi(r), B_\phi(r)$ and $\Pi(r)$.

Derive the linearized equations governing axisymmetric perturbations of this steady solution, with vertical wavenumber k and frequency ω. Obtain a single equation for the radial velocity perturbation or Lagrangian displacement. By quoting the properties of Sturm-Liouville systems, or otherwise, determine a necessary and sufficient condition for linear instability to axisymmetric perturbations.

Comment on the magnetic field strength required for this type of instability in a thin, Keplerian accretion disc. Discuss whether this instability is physically related to the axisymmetric magnetorotational instability in the presence of a vertical magnetic field.

3 A thin, differentially rotating disc is composed of an inviscid, incompressible fluid of uniform density ρ, surrounded by a vacuum. It experiences an axisymmetric gravitational potential $\Phi(r, z)$, where (r, ϕ, z) are cylindrical polar coordinates. The orbital, epicyclic and vertical frequencies are defined by

$$\Omega^2 = \frac{1}{r}\frac{\partial \Phi}{\partial r}\Big|_{z=0} ,$$
$$\kappa^2 = 4\Omega^2 + 2r\Omega\frac{d\Omega}{dr} ,$$
$$\Omega_z^2 = \frac{\partial^2 \Phi}{\partial z^2}\Big|_{z=0} .$$

Solve the equation of vertical hydrostatic equilibrium to determine the pressure distribution in the disc, in terms of ρ, Ω_z and the semi-thickness $H(r)$.

Consider axisymmetric linear waves of frequency ω and radial wavenumber $k(r)$ and satisfying $|kr| \gg 1$. Making appropriate approximations for a thin disc, show that the Eulerian pressure perturbation satisfies the equation

$$\frac{\partial^2 p'}{\partial z^2} = \left(\frac{\omega^2}{\omega^2 - \kappa^2}\right)k^2 p'.$$

Show further that the boundary condition

$$p' - \frac{\Omega_z^2}{\omega^2}\, z\, \frac{\partial p'}{\partial z} = 0$$

applies at $z = \pm H$.

By considering modes of even and odd symmetry, or otherwise, determine equations that describe all branches of the local dispersion relation of the disc. Find the limiting forms of the dispersion relation for large values of kH, and discuss the classification of modes in this limit. Sketch the dispersion relation in the quarter-plane $\omega, k \geqslant 0$, in the case $\Omega_z^2 > \kappa^2$.