UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2001 1.30 to 4.30

PAPER 41

PHYSICAL COSMOLOGY

*Answer any **THREE** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Friedman-Robertson-Walker metric is given by:

$$ds^2 = c^2 dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)],$$

where $a(t)$ is the cosmological scale factor.

(i) Use this metric to show that the spectral lines emitted at time t_e in a distant galaxy are redshifted by:

$$1 + z = \frac{1}{a(t_e)},$$

assuming that at the present time $a(t_0) = 1$.

(ii) Explain what is meant by luminosity distance d_L. Show that $d_L = (1+z)r_e$, where r_e is defined by $\int_{t_0}^{t_e} \frac{dt}{a(t)} \quad = \quad \int_0^{r_e} \frac{dr}{(1-kr^2)^{1/2}}$.

Derive d_L in the case of an Einstein-de Sitter universe ($\Omega = 1,\ \Lambda = 0$).

(iii) Describe briefly how Type Ia Supernovae can be used to constrain the geometry of the universe.

(iv) Consider a source at redshift z with luminosity $P(\nu) \propto \nu$, where ν is the frequency and the observed flux density $S(\nu_0)$ is:

$$S(\nu_0) = P(\nu_0)(1+z)^2 d_L^{-2}.$$

Assume a population of sources like the one above, with redshift independent co-moving source number density n_0, in an Einstein-de Sitter universe ($\Omega = 1,\ \Lambda = 0$). Show that the integrated background intensity at observed frequency ν_0 from this population of sources distributed out to very high redshift is:

$$I(\nu_0) \approx \frac{2c}{H_0} n_0 P(\nu_0),$$

where c is the speed of light and H_0 is the Hubble constant.

Explain what happens if the sources have $P(\nu) \propto \nu^2$.

2 In linear theory the evolution of the mass density contrast δ is given by:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G \rho_m \delta,$$

where $a(t)$ is the scale factor and ρ_m is the mean matter density.

(i) Discuss briefly the role of the second term on the left-hand side.

(ii) Solve this equation for the case of a static universe.

(iii) Find the solution of the above equation in the Einstein-de Sitter ($\Omega = 1,\ \Lambda = 0$) case, by taking into account the evolution of $a(t) \propto t^{2/3}$ and $\rho_m(t)$ in a matter-dominated universe.

(iv) If the dark matter of the universe were in the form of an unstable elementary particle that decays after the recombination epoch, it would presumably produce relativistic remnants, and the universe might re-enter a radiation dominated phase characterized by radiation density ρ_r.

Show that in the late-phase radiation era the Hubble parameter obeys

$$H(t) = \frac{1}{2t}.$$

Solve the equation of perturbations in this case, assuming that $\rho_m \ll \rho_r$.

Comment on this result compared with the solutions of sections (ii) and (iii).

3 (i) Define the two-point correlation function $\xi(r)$.

(ii) Derive the relation between the two-point mass correlation function $\xi(r)$ and the power spectrum of the mass fluctuations $P(k)$, defined by
$\langle \delta_{\vec{k}} \delta_{\vec{k}'} \rangle = (2\pi)^3 P(k) \delta_{Dirac}(\vec{k} + \vec{k}')$, where $\delta_{\vec{k}} = \int d^3r \delta(\vec{r}) e^{i\vec{k}\cdot\vec{r}}$. Express your result as a one-dimensional integral over k.

(iii) Assume that the mean number density of bright galaxies in the local universe is $\bar{n} = 0.01$ galaxies per $(h^{-1}\text{Mpc})^3$. Let their clustering be characterized by the two-point galaxy-galaxy correlation function $\xi(r) = (r/r_0)^{-\gamma}$ with $\gamma = 2$ and $r_0 = 5h^{-1}\text{Mpc}$.

Calculate the expected number of galaxies within a sphere of radius $10h^{-1}\text{Mpc}$ centred on a randomly chosen galaxy. Is this the same as the expected number of galaxies within a similar sphere about a random point in space?

(iv) Derive $\xi(r)$ for a power-spectrum of the form:

$$P(k) = Ak \qquad \text{for} \qquad 0 \leqslant k \leqslant k_{max},$$

and zero elsewhere, where A and k_{max} are constants.

(v) Consider a power-spectrum of the form

$$P(k) = Ak\,,$$

where A is a constant. By dimensional arguments show that in this case the gravitational potential fluctuations are the same on all scales.

Explain briefly how this model for mass density fluctuations on large scales can be constrained by measurements of the Cosmic Microwave Background on angular scales larger than about 1 degree.

4 (i) Discuss briefly the evidence for dark matter halos from rotation curves of galaxies.

(ii) Discuss briefly the evidence for dark matter in clusters of galaxies based on the velocity dispersion of cluster galaxies, the temperature of hot gas, and gravitational lensing.

(iii) Consider the evolution of a spherical proto-cluster in an Einstein-de Sitter ($\Omega = 1, \Lambda = 0$) universe. As the universe expands, the overdense region will expand more slowly compared to the background, will reach a maximum radius, contract and form a virialized cluster. Show that the radius of the virialized cluster is half the maximum radius.

(iv) Show, assuming hydrostatic equilibrium for a spherical cluster and an equation of state $P = \rho KT/(\mu m_p)$, that the cluster mass within radius r is

$$M(< r) = -\frac{KT(r)r}{\mu m_p G}\left[\frac{d\ln\rho}{d\ln r} + \frac{d\ln T}{d\ln r}\right].$$

(v) Describe how the total mass density parameter Ω can be estimated from the mass fraction of baryons in clusters, assuming that the baryonic contribution to Ω is known from Big Bang Nucleosynthesis. How does the result change if the abundance of Deuterium is observed to be smaller?