UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 1.30 to 4.30

PAPER 37

FORMATION, STRUCTURE AND EVOLUTION OF STARS

*Attempt **ANY NUMBER** of questions. The questions are of equal weight.*

Largely completed questions are strongly preferred to fragments.

The notation used is standard and you are reminded of the equations of stellar structure in the form:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\ ;\ \frac{dm}{dr} = 4\pi r^2\rho,$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi acr^2T^3}\ ;\ \frac{dL_r}{dr} = 4\pi r^2\rho\epsilon,$$

$$P = \frac{\Re\rho T}{\mu} + \frac{1}{3}aT^4.$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) A group of radiative zero-age main-sequence stars have opacity $\kappa = \kappa_o \rho^{\frac{1}{2}} T^{-5/2}$ and energy generation rate per unit mass $\epsilon = \epsilon_o \rho T^5$, with κ_o and ϵ_o constant, and are supported by gas pressure. Show that for these stars

$$R \propto M^{2/7} \qquad \text{and } L \propto M^{33/7}$$

Determine the slope of the main sequence in the theoretical HR diagram.

(b) A group of fully convective stars have surface opacity $\kappa = \kappa_1 \rho T^{10}$ and energy generation rate $\epsilon = \epsilon_o \rho T^5$ with κ_1 and ϵ_o constant and supported by gas pressure. Show that for these stars

$$R \propto M^{5/8} \qquad \text{and } L \propto M^2$$

Determine the slope of this main sequence in the theoretical HR diagram.

(c) Discuss briefly which parts of the main sequence might be described approximately by the groups of stars discussed above. What changes in the relevant physics will change the homology at higher and at lower masses?

2 A white dwarf has a helium core mass M, radius R and a thin hydrogen-rich, non-degenerate envelope of mass $M_{\rm env} \ll M$ and thickness $H \ll R$. Writing $z = r - R$, the surface density $\Sigma(z)$ in the envelope is defined by $d\Sigma = -\rho dz$ with $\Sigma = 0$ at $z = H$ and $\Sigma = \Sigma_o = M_{\rm env}/4\pi R^2$ at $z = 0$. Show that, to a good approximation, in the envelope $P = \Sigma g$ where $g = GM/R^2$.

Show that the approximate equations for the structure of the envelope are

$$\frac{dF}{d\Sigma} = -\epsilon \qquad \text{and} \qquad \frac{dT}{d\Sigma} = \frac{3\kappa F}{4acT^3},$$

where $F = L_r/4\pi R^2$.

The opacity is of the form $\kappa = \kappa_o \rho T^{-2}$ where κ_o is a constant. The white dwarf's luminosity is produced soley by hydrogen burning in the envelope. The hydrogen burns steadily, mainly at the base of the envelope, with $\epsilon = \epsilon_o \rho T^{15}$, where ϵ_o is a constant. Radiation pressure can be neglected. Setting $y = T^7$ and $x = \frac{1}{2}\Sigma^2$ show that the structure equations can be written in the form

$$\frac{d^2y}{dx^2} = -\omega^2 y^2,$$

where ω^2 is a positive constant.

Show that appropriate boundary conditions are:

$$at \quad x = 0,\ y = 0 \text{ and } \frac{dy}{dx} = \frac{AL}{4\pi R^2},$$

where A is a constant;

$$at \quad x = \frac{1}{2}\Sigma_0^2,\ y = T_0^7 \text{ and } \frac{dy}{dx} = 0.$$

By multiplying both sides by $\frac{dy}{dx}$ and integrating once, deduce that the temperature at the base of the envelope satisfies

$$T_o \propto \left(\frac{L}{4\pi R^2}\right)^{2/21}.$$

Integrate again, applying the boundary conditions and noting that $\int_0^1 \frac{dy}{(1-y^3)1/2}$ is a constant, to derive the relationship

$$\frac{L}{4\pi R^2} \propto \left(\frac{M_{\rm env}}{4\pi R^2}\right)^{-6}.$$

Comment on the stability of hydrogen burning in a white dwarf envelope.

3 In an Algol system a red giant of mass M_1, fills its Roche lobe of effective radius R_L and is transferring mass on to a main-sequence companion of mass M_2. The structure of the giant is such that its radius

$$R = f(L) M_1^{-0.27},$$

where $f(L)$ is a function of luminosity only. Luminosity is itself a function only of conditions deep in the star and is independent of M_1.

The effective Roche lobe radius is given by

$$\frac{R_L}{a} = 0.462 \left(\frac{M_1}{M} \right)^{1/3},$$

where $M = M_1 + M_2$ and a is the orbital separation. When the spin of the stars can be neglected show that the total angular momentum of the system is

$$J = \frac{M_1 M_2}{M} a^2 \Omega.$$

Thence show that conservative mass transfer ($\dot{M}_2 = -\dot{M}_1, \dot{J} = 0$) is dynamically stable only if the mass ratio

$$q = \frac{M_1}{M_2} < q_{\text{crit}} \approx 0.7$$

One way for the system to have reached this state is for star 1 to fill its Roche lobe before reaching the giant branch. Explain why this is so.

The radius of a star of mass M_1, at the base of the giant branch is given by

$$\left(\frac{R_{\text{BGB}}}{R_\odot} \right) = 1.68 \left(\frac{M_1}{M_\odot} \right)^{5/3}.$$

Show that if star 1 is to fill its Roche lobe before becoming a giant then the initial orbital period P must satisfy:

$$P < P_o \left(\frac{M_1}{M_\odot} \right)^2.$$

You may take $P_o = 0.8$ days for the remainder of this question.

If the mass transfer is conservative show that

$$P (M_1 M_2)^3 = \text{const.}$$

Hence show that P/M_1^2 would have had a minimum when $M_1 = \frac{5}{8} M$.

Algol now has a period of 3 days and masses $M_1 = 1 M_\odot, M_2 = 3 M_\odot$. Argue that it could well have formed via conservative case A or early B mass transfer. Another semi-detached system with a mass-losing giant, RT Lac, has $P = 5.1$ days, $M_1 = 0.9M_\odot$ and $M_2 = 1.5M_\odot$. Show that it could not have formed conservatively.

Suggest how RT Lac might have formed.

4 Give a description of the physical processes which need to be taken into account in determining the Rosseland mean opacity $\kappa(\rho, T)$ in the density and temperature region relevant to stellar structure.

or

Write brief essays on three of the following topics:

(a) Stellar mass determination

(b) The evidence that stars are powered by nuclear reactions

(c) Thermal pulses in Asymptotic Giant Branch stars

(d) The end points of stellar evolution