UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2001 9 to 11

PAPER 36

MAGNETOHYDRODYNAMICS

*At least **TWO** questions should be attempted.*

Complete answers are preferred to fragments.

The three questions are of equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Incompressible fluid of density ρ, magnetic diffusivity η, and kinematic viscosity ν, occupies the half-space $y > 0$ and is bounded by a fixed plane, rigid, insulating boundary $y = 0$. The region $y < 0$ is insulating. A line current $J \cos \omega t$ flows in a wire parallel to the z-axis and located at $x = 0$, $y = -b$ $(b > 0)$. This produces a magnetic field $\mathbf{B}(x, y)e^{i\omega t}$ (real part understood) and the resulting velocity field in the fluid is $\mathbf{u}(x, y, t)$.

(a) Write down the equations and boundary conditions which (in principle) determine the fields $\mathbf{u}$ and $\mathbf{B}$.

(b) Suppose that the frequency ω is large, so that $(\eta/\omega)^{1/2} \ll b$. By assuming that ω is in effect infinite so that the fluid behaves like a perfect conductor, show that, in a first approximation,

$$\mathbf{B}(x, 0-) = -\frac{\mu_0 J}{\pi} \frac{b}{x^2 + b^2}\, \mathbf{e}_x\, , \tag{1}$$

where $\mathbf{e}_x = (1, 0, 0)$ and μ_0 is the constant 'permeability of free space'.

[*Hint: place an image current at* $(0, b)$.]

(c) Now obtain $\mathbf{B}(x, y)$ in $y > 0$, adopting (1) as a boundary condition, and neglecting the motion of the fluid. Show that the mean Lorentz force (averaged over a time-period $2\pi/\omega$) is

$$\mathbf{F}(x, y) = (2\mu_0\delta)^{-1}(\mathbf{B}(x, 0))^2 e^{-2y/\delta}\mathbf{e}_y\, ,$$

where $\delta = (2\eta/\omega)^{1/2}$, and $\mathbf{e}_y = (0, 1, 0)$.

(d) Assuming that the (similarly averaged) velocity field in the region near the boundary (i.e. for $y = O(\delta)$) is $\mathbf{u} = (u(x, y), v(x, y), 0)$, where $|v| \ll |u|$, find $u(x, y)$, and hence obtain an effective boundary condition for the bulk flow of the fluid. Sketch the streamlines that you would expect for this flow.

2 (a) Explain the principles of *mean-field electrodynamics* by which an 'alpha-effect' and a 'turbulent diffusivity effect' may be obtained.

(b) By considering an example in which the velocity field consists of three circularly polarised waves propagating in mutually perpendicular directions, show how the coefficient α of the alpha-effect can be related to the mean helicity of the flow.

(c) Show that the mean-field equation with constant isotropic α and turbulent diffusivity $\beta(> 0)$ admits exponentially growing modes of Beltrami structure.

(d) Discuss the implications of this theory for planetary and cosmical electrodynamics.

3 An incompressible fluid, which is perfectly conducting but nevertheless endowed with kinematic viscosity $\nu > 0$, is contained in a domain $\mathcal{D}$ with fixed closed boundary $\partial\mathcal{D}$. The magnetic field $\mathbf{B}(\mathbf{x}, t)$ in $\mathcal{D}$ satisfies $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\mathcal{D}$, where $\mathbf{n}$ is the unit outward normal. The fluid moves with velocity $\mathbf{v}(\mathbf{x}, t)$ where $\mathbf{v} = 0$ on $\partial\mathcal{D}$.

(a) Define the total magnetic helicity $\mathcal{H}_M$ of the field, and prove that this is constant. Give a physical interpretation of this result.

(b) Suppose that at time $t = 0$, $\mathbf{v}(\mathbf{x}, 0) = 0, \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x})$, where $\mathbf{B}_0(\mathbf{x})$ has non-zero helicity. Obtain the energy equation in the subsequent evolution, and show that, assuming that the vorticity field remains smooth, the field ultimately settles down to a state of magnetostatic equilibrium.

(c) Explain why current sheets may be expected to appear in this relaxation process.

(d) Explain how such current sheets can appear if the domain $\mathcal{D}$ and the field $\mathbf{B}_0(\mathbf{x})$ are two-dimensional.