UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2001 1.30 to 4.30

PAPER 35

ASTROPHYSICAL FLUID DYNAMICS

Attempt ***THREE*** *questions. The questions are of equal weight.*

Candidates may bring their notebooks into the examination.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A cosmological constant $\Lambda > 0$ may be incorporated into Newtonian cosmology by writing Poisson's equation for the gravitational potential Φ as

$$\nabla^2 \Phi + \Lambda = 4\pi G\rho.$$

Derive the equation for the expansion factor $R(t)$ describing the evolution of an infinite uniform universe, and show that it has a first integral

$$\dot{R}^2 = \frac{8\pi G\rho_0}{3R} + \frac{1}{3}\Lambda R^2 - \kappa,$$

where ρ_0 is the density at $t = t_0$, when $R = 1$. Show that at large times an expanding universe expands exponentially with time.

Consider now the adiabatic development of an infinitesimal perturbation ρ' in density of the form

$$\rho'(\mathbf{r}, t) = \varepsilon(t)\rho(t)e^{iR^{-1}\mathbf{k}\cdot\mathbf{r}},$$

where $\mathbf{k}$ is a constant vector. Assume the pressure to satisfy $p \propto \rho^\gamma$, with $\gamma = \frac{4}{3}$, in both the unperturbed and the perturbed universe. Show that if $\Lambda \gg |\kappa|$ and $8\pi G\rho_o \ll \Lambda$, ε satisfies

$$\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2} + 2\mu\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \eta^2 e^{-3\mu(t-t_0)}\varepsilon = 0, \qquad (\star)$$

where μ is a constant, depending on Λ, and $\eta^2 = c_0^2 k^2 - 4\pi G\rho_0$, in which c_0 is the adiabatic sound speed at $t = t_0$.

Discuss the solution for small and large times $t - t_0$, both when $\eta^2 < 0$ and when $\eta^2 > 0$, as follows: Reduce equation $(\star)$ to standard form (having no first derivative of the dependent variable), and then write down solutions in the JWKB approximation and discuss their properties. You may find it useful to evaluate the integral in the case when $\eta^2 < 0$. Show that a general perturbation tends to a constant as $t \to \infty$, whatever the values of η^2 and Λ. Show also that the constant to which ϵ tends at large times is greater than its value at $t = t_0$ if $c_0^2 k^2 < 4\pi G\rho_0$.

2 A star of initial radius $R_\star$ explodes into the surrounding uniform interstellar medium of perfect gas with density ρ_0 and adiabatic exponent γ, liberating energy E. Explain carefully why the radius $R(t)$ of the strong shock that is formed, when $R \gg R_\star$, satisfies

$$R \simeq \xi_0 \left(\frac{E}{\rho_0}\right)^{\frac{1}{5}} t^{\frac{2}{5}},$$

where ξ_0 is a dimensionless constant and t is time. Write down expressions for the density, ρ_1, pressure, p_1, and velocity, u, immediately behind the shock in terms of ρ_0 and the velocity V of the shock.

In order to estimate ξ_0, consider the mass swept up by the shock to be confined to a thin shell of density ρ_1 and thickness d at pressure p_1, and regard the pressure $p_\mathrm{c} = \alpha p_1$ interior to the shell to be uniform. By considering the momentum balance of the shell, establish the relation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = KR^{3(\alpha-1)},$$

where K is a constant.

The total energy of the system may be regarded as being the sum of the internal and the kinetic energy of the shell, and the internal energy of the hot diffuse gas internal to the shell. Deduce the value of α, and show that

$$\xi_0 = \left[\frac{75}{16\pi}\frac{(\gamma-1)(\gamma+1)^2}{5\gamma-3}\right]^{\frac{1}{5}}.$$

Estimate the kinetic energy of the material interior to the shell, and compare it with E.

Why is this calculation relevant to astrophysics? What are its limitations?

3 A high-order acoustic mode of frequency ω in a spherically symmetrical star of radius R in which the sound speed is $c(r)$ can be regarded as a superposition of resonant waves satisfying the dispersion relation $\omega = kc$, where k is the magnitude of a local wavenumber. Explain briefly why the resonance condition (eigenvalue equation) can be written approximately

$$\int \left(\frac{\omega^2}{c^2} - \frac{L^2}{r^2}\right)^{\frac{1}{2}} \mathrm{d}r = (n+\alpha)\pi,$$

where n, L and α are constants. Indicate how, for a given set of mode frequencies, one can attempt to invert this equation to determine the sound speed between the highest and lowest of the turning points r_{t}, where $a(r_{\mathrm{t}}) = w, a = c/r$ and $w = \omega/L$.

Describe briefly how from observations of the Sun one can determine n, L and ω for each mode, and the constant α.

Observations of the acoustic modes of solar oscillation of intermediate and high degree l are found to satisfy

$$\frac{(n+3/2)\pi}{\omega} = \frac{w}{w_0}\left(1 - \frac{3w^2}{2\pi w_0^2}\right) \equiv F(w), \qquad w_{\mathrm{s}} \leqslant w \leqslant w_1$$

where w_0 is a constant and $w_1 > \sqrt{\pi} w_0/3$. Obtain the relation

$$\frac{r}{r_{\mathrm{s}}} = \exp\left\{\frac{-\frac{2a^2}{\pi w_0^2}\int_{x_{\mathrm{s}}}^{1} x(1 - 9a^2x^2/2\pi w_0^2)\,\mathrm{d}x}{\sqrt{1-x^2}}\right\}$$

in which $x_{\mathrm{s}}(r) = w_{\mathrm{s}}/a$ and $a(r_{\mathrm{s}}) = w_{\mathrm{s}}$. Using the substitution $x = \sin\theta$, or otherwise, obtain the expression

$$c(r) = \frac{\sqrt{\pi}}{3}\left[1 - \sqrt{1 - \ln(\frac{r_{\mathrm{s}}}{r})}\right]^{\frac{1}{2}} w_0 r.$$

If the maximum value of l in the mode set is large enough for your formula to be valid close to the surface of the Sun, confirm that your inversion is consistent with the surface layers of the Sun being a plane-parallel polytrope of index $\mu = 3$ and adiabatic exponent γ, by showing that $c \simeq c_0 z^{\frac{1}{2}}$, where $z = R - r \ll R$. Determine the constant w_0 in terms of R, γ and the mass M of the Sun. Hence show that the value of the sound speed at $r = e^{-1}R$ is $\frac{\pi}{3}\sqrt{\gamma GM/2R}$.

4 Write an essay on the fluid dynamics of stellar rotation, with particular emphasis on the Sun. Discuss the mechanisms that might cause fluid flow, and consider the importance of the timescale of the mechanisms you discuss.

[*You may find the following integrals useful*:

$$\int \sec\theta \, \mathrm{d}\theta = \ln(\sec\theta + \tan\theta),$$

$$\int \mathrm{cosec}\theta \, \mathrm{d}\theta = -\ln(\mathrm{cosec}\theta + \cot\theta),$$

$$\int \frac{\mathrm{d}\theta}{\sinh\theta} = \frac{1}{2}\ln\left(\frac{\cosh\theta - 1}{\cosh\theta + 1}\right).$$]