UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2001 1.30 to 3.30

PAPER 33

TIME SERIES AND MONTE CARLO INFERENCE

*Attempt any **THREE** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Monte Carlo Inference

(a) Let

$$h(x) = \frac{f(x)}{\int_{-\infty}^{\infty} f(y)dy}$$

denote a univariate density, with

$$\sup_{x\in\mathbb{R}} \left(\frac{f(x)}{g(x)}\right) = M \in \mathbb{R},$$

for some density g, defined on $\mathbb{R}$.

(i) Describe the rejection sampling algorithm for generating observations from h using a set of observations from g.

(ii) Suppose that we already have a sample of n observations from g. Calculate the probability that the rejection algorithm accepts an observation at any stage and hence show that the expected size of the resulting sample from h is given by

$$nM^{-1} \int_{-\infty}^{\infty} f(y)dy.$$

(b) Suppose that h is the half-Normal density, given by

$$h(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2}, \quad x \geqslant 0,$$

and that we are able to gain samples from a density g where

$$g(x) = \lambda e^{-\lambda x}, \quad \lambda > 0, x \geqslant 0.$$

Setting the $f(x) = h(x)$, show that the value of M is given by

$$M = \sqrt{\frac{2e^{\lambda^2}}{\pi\lambda^2}}.$$

Hence suggest how the rejection sampling algorithm might be used to sample from the standard $N(0,1)$ distribution.

2 Monte Carlo Inference

(i) Define the Metropolis Hastings algorithm and the Gibbs sampler, for obtaining a dependent sample from some distribution $\pi(\mathbf{X}), \mathbf{X} \in \mathbb{R}^k$.

(ii) Now take the bivariate Normal distribution, where $k = 2$ and

$$\pi(\mathbf{x}) = \frac{1}{2\pi|\mathbf{\Sigma}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right\},$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and } \mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Find the form of the full conditional distributions $\pi(x_1|x_2)$ and $\pi(x_2|x_1)$. Hence illustrate how the Gibbs Sampler can be used to obtain a dependent sample from the bivariate Normal distribution.

(iii) By taking the proposal density as the bivariate Normal centred at $\mathbf{x}$ and with covariance equal to the identity matrix, calculate the Metropolis Hastings acceptance probability. Hence, show how we might sample from the bivariate Normal via the Metropolis Hastings algorithm.

3 Time Series

Suppose $\{\epsilon_t\}$ is Gaussian white noise. For given $\theta_1, \ldots, \theta_q$, let $\{x_t\}$ be the stationary process given by

$$x_t = \frac{5}{6}x_{t-1} - \frac{1}{6}x_{t-2} + \epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_q\epsilon_{t-q}\,,$$

and let $\{\gamma_k\}$ be its autocovariance function. Show that $\gamma_k - \frac{5}{6}\gamma_{k-1} + \frac{1}{6}\gamma_{k-2} = 0$ for all $k > q$.

Use this to show that there exists $B < \infty$ such that $|\gamma_k| < (1/2)^k B$ for all k.

Let j, m and N be integers, with $j \leqslant m$ and N odd. Let $\omega = 2\pi j/(2m+1)$ and $T = (2m+1)N$. Define

$$A = \frac{1}{\sqrt{\pi T}} \sum_{t=1}^{T} x_t \cos(\omega t) \quad \text{and} \quad B = \frac{1}{\sqrt{\pi T}} \sum_{t=1}^{T} x_t \sin(\omega t)\,.$$

Show that

$$\mathbb{E}AB = \frac{1}{\pi T}\left[\gamma_0 \sum_{t=1}^{T} \cos(\omega t)\sin(\omega t) \right.$$
$$\left. + \sum_{k=1}^{T-1} \gamma_k \left(\sum_{t=1}^{T-k} \cos(\omega t)\sin(\omega(t+k)) + \sum_{t=k+1}^{T} \cos(\omega t)\sin(\omega(t-k)) \right) \right].$$

Deduce that

$$|\mathbb{E}AB| \leqslant \frac{1}{\pi T} \sum_{k=1}^{T-1} 2k|\gamma_k|\,.$$

Assuming that the variances of A and B also converge as $N \to \infty$, deduce that the joint distribution of A and B converges to that of two independent Gaussian random variables.

What are the limits of $\mathbb{E}A^2$ and $\mathbb{E}B^2$ as $N \to \infty$?

Discuss the asymptotic unbiasedness and consistency of $I(\omega) = A^2 + B^2$ as an estimator of the value of the spectral density function at ω.

You may use the facts that

$$\sin(\omega(t+k)) + \sin(\omega(t-k)) = 2\sin(\omega t)\cos(\omega k)$$

$$\cos(\omega(t+k)) + \cos(\omega(t-k)) = 2\cos(\omega t)\cos(\omega k)$$

$$\sum_{t=1}^{T} \cos(\omega t)\sin(\omega t) = 0\,, \quad \sum_{t=1}^{T} \sin^2(\omega t) = \sum_{t=1}^{T} \cos^2(\omega t) = T/2.$$

4 Time Series

Consider the ARMA(1,1) model $x_t = \phi x_{t-1} + \epsilon_t + \theta\epsilon_{t-1}$, where $\{\epsilon_t\}$ is Gaussian white noise with variance σ^2. Define $S_t = (x_{t-1}, \epsilon_t, \epsilon_{t-1})^\top$ and $w_t = (0, \epsilon_t, 0)^\top$. Find G and F such that $x_t = FS_t$ and $S_t = GS_{t-1} + w_t$.

Assume that the distribution of S_t given $x_1, \ldots, x_t$ is multivariate normal $N(\hat{S}_t, P_t)$, $t \geqslant 1$. Describe in general terms the main features of the Kalman filter, as it is used to determine $(\hat{S}_t, P_t)$ from $\hat{S}_0$, P_0 and $x_1, \ldots, x_t$. You are not required to give any detailed formulae.

How might one choose $\hat{S}_0$ and P_0?

Show that the problem of determining the maximum likelihood estimators of ϕ, θ and σ^2, given $x_1, \ldots, x_T$, is equivalent to minimizing with respect to these parameters

$$\sum_{t=1}^{T} \left[\log(2\pi) + \log V_t + \frac{(x_t - \hat{x}_t)^2}{V_t} \right] .$$

where $\hat{x}_t = FG\hat{S}_{t-1}$ and $V_t = FGP_{t-1}G^\top F^\top$.

Consider the case of AR(1). Assume x_1 has the stationary distribution of this process. What are $V_1, \ldots, V_T$?

Show that in this case that maximum likelihood estimator of ϕ is approximately

$$\hat{\phi} = \sum_{t=2}^{T} x_t x_{t-1} \bigg/ \sum_{t=2}^{T} x_{t-1}^2 \ .$$