UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 1.30 4.30

PAPER 34

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt **FOUR** questions*

*There are **six** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on bargaining. Your account should include a description of the terms jointly dominated, Pareto optimal, bargaining (or negotiation) set, Nash arbitration procedure and maximin bargaining solution.

2 Consider the optimization problem

$$\min f(x)$$

subject to $h(x) = b,\ x \in X \subset \mathbb{R}^n, b \in \mathbb{R}^m$. Define the Lagrangian function for this problem and then state and prove the Lagrangian Sufficiency Theorem. Define the function ϕ by

$$\phi(b) = \inf_{x \in X \,:\, h(x) = b.} f(x)$$

Define the Strong Lagrangian property and show that the following are equivalent

(1) there exists a non-vertical supporting hyperplane to ϕ at b

(2) the problem is Strong Lagrangian.

A company is planning to spend £a on advertising. It costs £3,000 per minute to advertise on television and £1,000 per minute to advertise on radio. If the company buys x minutes of television advertising and y minutes of radio advertising, its revenue in thousands of pounds is given by $f(x, y) = -2x^2 - y^2 + xy + 8x + 3y$. How can the company maximise its revenue? Compare the increase in revenue for each additional advertising pound when a = 1,000 with the case when a = 10,000.

3 Consider the general class of linear programmes given by

$$\min \quad c^T x$$

$$\text{subject to } Ax = b,\ x \geq 0$$

where $x \in \mathbb{R}^n, b \in \mathbb{R}^m$ and where all the entries in A, b and c have absolute magnitudes bounded by $U < \infty$.

Show that such linear programmes can be reduced to the special form

$$\begin{aligned} \min\ & c^{*T} y \\ \text{subject to } & A^* y = 0 \\ & 1^T y = 1 \\ & y \geq 0 \end{aligned}$$

with the additional properties that

(i) $y^{(0)} = (1/n^*, \ldots, 1/n^*)^T$ is feasible (where $y \in \mathbb{R}^{n^*}$)

(ii) the optimal value of the objective is zero.

Why is this a useful result?

4 Consider a network with n nodes and set of arcs A. Let $c_{ij} > 0$ for $(i, j) \in A$ be the length of arc (i, j) and set $c_{ij} = \infty$ if $(i, j) \notin A$. Regarding n as the root node, define the all-to-one shortest path problem. Define the Bellman-Ford algorithm for solving this problem. Discuss why this is referred to as a label-correcting algorithm.

Define v_i to be the shortest path length from node i to node n. Suppose that $j \neq n$ is a node such that $c_{jn} = \min_{i \neq n} c_{in}$. Show that $v_j = c_{jn}$ and $v_j \leq v_k$ for all nodes $k \neq n$. Define Dijkstra's algorithm for the all-to-one shortest path problem. Discuss why this is referred to as a label-setting algorithm. Apply Dijkstra's algorithm to the following network with root node $n = 4$,

where the numbers beside the arcs denote the arc's length.

5 The payoff matrix for a two-person non-zero sum game is

$$\begin{array}{c} \\ I_1 \\ I_2 \end{array} \begin{array}{c} \quad II_1 \qquad\quad II_2 \\ \begin{pmatrix} (3, & 8) & (4, & 4) \\ (2, & 0) & (0, & 6) \end{pmatrix} \end{array}$$

Find all equilibrium pairs when considered as a non-cooperative game. Then find the maximin bargaining solution when the game is considered as a cooperative game. Which game would II prefer to play?

6 Consider the game with characteristic function $v(1) = 1$, $v(2) = 2$, $v(3) = 3$, $v(1,2) = 3$, $v(1,3) = 10$, $v(2,3) = 6$ and $v(1,2,3) = 12$.

Define

(a) the set of imputations

(b) the core

(c) the nucleolus

and find them for the game defined above.