

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2001 9 to 12

PAPER 22

ADVANCED FINANCIAL MODELS

*Attempt **FOUR** questions. The questions carry equal weight.*

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

1 Write an essay on optimal hedging in the least-squares sense in a one-period financial model. Your essay should cover the notions of attainable claims, dominated and equivalent martingale measures, the minimal martingale measure and a proof of the fact that the model is complete if and only if there is a unique dominated martingale measure.

2 ‘The value of any portfolio is the conditional expected value under any martingale measure of the sum of its future discounted dividends.’

Write an essay which explains this statement in the context of a multiperiod, discrete-time market.

3 State and prove Girsanov’s Theorem. Describe briefly the relevance of this result to the pricing of contingent claims in the Black-Scholes model.

Let $W_t^\mu = W_t + \mu t$ denote a standard Brownian motion with linear drift and set $M_t^\mu = \sup_{0 \leq s \leq t} W_s^\mu$. Prove that for $a > 0$ and $x \leq a$,

$$\mathbb{P}(W_t^\mu \leq x, M_t^\mu < a) = \Phi\left(\frac{x - \mu t}{\sqrt{t}}\right) - e^{2a\mu} \Phi\left(\frac{x - 2a - \mu t}{\sqrt{t}}\right).$$

Consider an option that pays 1 at time t_0 if the stock price $\{S_t\}$ reaches a fixed level $b > S_0$ at some time in the interval $[0, t_0]$; otherwise, it pays nothing. In the context of the Black-Scholes model, derive an expression for the price at time 0 of this option.

4 Suppose that in the Black-Scholes model, the stock price at time t is S_t , the fixed interest rate is ρ and the volatility is σ . Show that $f(S_t, t)$ is the value of a self-financing portfolio if and only if the function $f = f(x, t)$ satisfies

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \rho x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} - \rho f = 0.$$

Determine the price at time $t < t_0$ of a European-style claim that pays $C = S_{t_0} \ln(S_{t_0})$ at time t_0 . What is the holding in stock at time t in the portfolio that replicates this claim? Justify your answer.

5 **Either:** Write an essay on determining optimal investment strategies for an investor in the Black-Scholes model who seeks to maximize the expected utility of his final wealth at time $t_0 > 0$ for given initial wealth at time 0.

or: Write an essay on pricing path-dependent contingent claims in the Black-Scholes model.

- 6 Write an essay on the modelling of interest rates using Gaussian random fields.