UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2001 9 to 11

PAPER 20

DIOPHANTINE ANALYSIS AND TRANSCENDENCE THEORY

Answer any ***TWO*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove that the numbers e and π are transcendental. State the more general Lindemann theorem and show that it implies that $\sin\alpha/\sin\beta$ is transcendental for non-zero algebraic α, β with $\alpha \neq \pm\beta$.

2 EITHER use transcendence methods to establish the Riemann hypothesis for elliptic curves over finite fields with q elements, where q is an odd prime, OR use Mahler's method to show that the series $\sum_{n=0}^{\infty} z^{l^n}$, where l is an integer ≥ 2, assumes transcendental values for all algebraic z with $0 < |z| < 1$.

Show that, if $l \geq 3$ and if $z = p/q$ where p, q are positive integers with $p < q^\delta$ with $\delta < \frac{1}{3}$, then the transcendence of the latter series can be proved from the Thue-Siegel-Roth theorem.

3 (i) Outline a proof of the Gelfond-Schneider theorem.

(ii) State the Schneider-Lang theorem. By considering the meromorphic functions z and e^z, establish the transcendence of e^α for algebraic $\alpha \neq 0$.

4 (i) State and prove Siegel's lemma on the solution of linear equations.

(ii) Define Siegel's E-functions. Show that sums of E-functions are again E-functions. State the Siegel-Shidlovsky theorem and indicate the particular E-functions that give Lindemann's theorem.

5 Assuming an appropriate estimate for a logarithmic form, show that the equation $\alpha x + \beta y = 1$, where α, β are non-zero elements of an algebraic number field K, has only finitely many solutions in units x, y in K. Indicate how this leads EITHER to a proof of Thue's theorem on the equation $F(x, y) = m$ OR to a treatment of the hyperelliptic equation $y^2 = f(x)$.

6 (i) Show how the exponential equation $ax^n - by^n = c$ can be effectively solved by the theory of linear forms in logarithms. Indicate how one proves from a generalization of this result that there are only finitely many exponents m for which the superelliptic equation $y^m = f(x)$ is soluble.

(ii) State the *abc*-conjecture. Show EITHER that it implies that the Fermat equation $x^n + y^n = z^n$, with $n > 3$, has only finitely many solutions OR that it implies the same for the Catalan equation $x^p - y^q = 1$, with $p > 2, q > 2$.

7 Write an essay on the theory of logarithmic forms, outlining EITHER the way the basic estimates are established OR how they are applied to the practical solution of Diophantine equations.