UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 1.30 to 4.30

PAPER 2

NOETHERIAN ALGEBRAS

*Answer any **THREE** questions.*

All questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let k be a field and G be the group of upper unitriangular integral 3×3 matrices

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \ : \ a, b, c \in \mathbb{Z} \right\}$$

(i) By proving an appropriate version of Hilbert's Basis Theorem, show that kG is both left and right Noetherian.

(ii) Prove that if r and $s \in kG$ satisfy $rs = 0$, then $r = 0$ or $s = 0$.

(iii) State and sketch the proof of Ore's Theorem, and deduce that there is a division ring which is both a left and a right classical ring of quotients of kG.

2 Write an essay about the module theory for Weyl algebras. This should include a discussion of the characteristic variety of a module, and a sketch proof of Bernstein's inequality.

3 (i) Let $S = \bigoplus_{i=0}^{\infty} S_i$ be a positively graded commutative ring.

Define the Poincaré series of a finitely generated graded S-module $N = \bigoplus_{i=0}^{\infty} N_i$ with respect to an integral-valued additive function λ on finitely generated S_0-modules.

State and prove the Hilbert-Serre Theorem.

(ii) Let R be the enveloping algebra of a finite dimensional complex Lie algebra $\mathfrak{g}$ with $dim_{\mathbb{C}}\mathfrak{g} = 3$.

Show how R may be positively filtered so that the associated graded ring is isomorphic to a polynomial algebra $\mathbb{C}[X_1, X_2, X_3]$.

Define the dimension of a finitely generated R-module M and show that it is at most 3.

4 (i) Let R be a commutative Noetherian ring and I be an ideal of R. Let N be a submodule of a finitely generated R-module M.

Show that there exists $k \geqslant 0$ such that

$$N \cap I^i M = I^{i-k}(N \cap I^k M) \quad \text{for} \quad i \geqslant k$$

(ii) Let R be a complete (not necessarily commutative) negatively filtered ring whose associated graded ring grR is left Noetherian. Show that R is left Noetherian.

(iii) Define a uniform pro-p group and sketch a proof that, for $p \geqslant 3$, the kernel of the canonical map $GL_p(\mathbb{Z}_p) \longrightarrow GL_p(\mathbb{F}_p)$ is a uniform pro-p group.

(iv) Let G be the additive group of $\mathbb{Z}_p$ and let I be the kernel of the $\mathbb{F}_p$-linear map

$$\mathbb{F}_p G \longrightarrow \mathbb{F}_p$$

$$\sum \lambda_g g \longrightarrow \sum \lambda_g$$

Show that the associated graded ring of the I-adic filtration of $\mathbb{F}_p G$ is isomorphic to the polynomial algebra $\mathbb{F}_p[X]$.