UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2001 1.30 to 4.30

# PAPER 17

# CATEGORY THEORY

*Attempt **SIX** questions. The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction, $F \dashv G$.

(i) Define the *unit* $\eta$ and the *counit* $\varepsilon$ of the adjunction.

(ii) Prove the triangle identities, $(\varepsilon F)\circ(F\eta) = 1_F$ and $(G\varepsilon)\circ(\eta G) = 1_G$.

(iii) Prove that given functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and natural transformations $1 \xrightarrow{\eta} GF$, $FG \xrightarrow{\varepsilon} 1$ satisfying the triangle identities, there is a unique adjunction between $F$ and $G$ with $\eta$ as its unit and $\varepsilon$ as its counit.

**2** (i) Fix a nonempty topological space $S$, and let $\mathcal{O}(S)$ be the poset of open subsets of $S$, ordered by inclusion. Let

$$\Delta : \mathbf{Set} \longrightarrow [\mathcal{O}(S)^{\mathrm{op}}, \mathbf{Set}]$$

be the functor assigning to a set $A$ the presheaf $\Delta A$ with constant value $A$. Exhibit a chain of adjoints

$$\Lambda \dashv \Pi \dashv \Delta \dashv \Gamma \dashv \nabla.$$

(ii) Let $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor taking a small category to its set of objects. Exhibit a chain of adjoints

$$C \dashv D \dashv O \dashv I.$$

(iii) Do either of these chains of adjoints extend further in either direction?

(*In parts (i) and (ii), when you define a functor you are only required to describe its effect on objects, and when you show adjointness you are not required to carry out any formal checks of naturality.*)

**3**

State and prove the Yoneda Lemma. Deduce:

(i) that the Yoneda embedding is full and faithful

(ii) that for objects $A, B$ of a locally small category $\mathcal{C}$, $A \cong B$ if and only if $\mathcal{C}(C, A) \cong \mathcal{C}(C, B)$ naturally in $C \in \mathcal{C}$

(iii) that a functor $X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable if and only if there exist an object $A \in \mathcal{C}$ and an element $u \in X(A)$ which is 'universal' in a sense you should make precise.

**4** (i) Show that if a category has a terminal object, all binary products and all equalizers, then it has all finite limits.

(ii) Let $\mathcal{C}$ be a category with all finite limits and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor which preserves finite products and equalizers. Show that $F$ preserves all finite limits.

(iii) Deduce from (i) that if a category has a terminal object and all pullbacks then it has all finite limits.

**5**

Let $\mathbb{C}$ be a small category and $\mathcal{S}$ a complete category.

(i) Show that the functor category $[\mathbb{C}, \mathcal{S}]$ is complete, and that for each object $C$ of $\mathbb{C}$ the evaluation functor

$$\begin{aligned} \mathrm{ev}_C : [\mathbb{C}, \mathcal{S}] &\longrightarrow \mathcal{S} \\ X &\longmapsto X(C) \end{aligned}$$

preserves limits.

(ii) Characterize the epics in the category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on $\mathbb{C}$, justifying your answer.

**6**

Let $\mathbb{C}$ be a small category.

(i) Prove that any presheaf $X \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is a colimit of representable presheaves.

(ii) What does it mean for a category to be *cartesian closed*? Show that $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed. (You may assume that products exist and are computed pointwise in a presheaf category.)

(iii) Now suppose that $\mathbb{C}$ is cartesian closed. Prove that the Yoneda embedding $H_\bullet$ preserves exponentials.

**7**

Consider the following three conditions on a functor $U$ from a locally small category $\mathcal{C}$ to **Set**:

**A.** $U$ has a left adjoint

**R.** $U$ is representable

**L.** $U$ preserves limits.

(i) Show that $\mathbf{A} \Rightarrow \mathbf{R} \Rightarrow \mathbf{L}$.

(ii) Show that if $\mathcal{C}$ has small coproducts then $\mathbf{R} \Rightarrow \mathbf{A}$.

(iii) Show that if $\mathcal{C}$ is complete, well-powered and has a coseparating set then the three conditions are equivalent.

**8**

State the General Adjoint Functor Theorem and the Special Adjoint Functor Theorem.

**Either** show that if a complete, locally small category has a weakly initial set of objects then it has an initial object, and explain in outline how this result leads to a proof of the General Adjoint Functor Theorem

**Or** assuming the General Adjoint Functor Theorem, prove the Special Adjoint Functor Theorem.

**9** (i) What is a *monad* on a category? What is the *category of algebras* for a monad? What does it mean for a functor to be *monadic*?

(ii) Explain how an adjunction gives rise to a monad, and explain briefly why every monad arises in this way.

(iii) Prove that a monadic functor creates limits.

**10**

State the Monadicity Theorem, and sketch a proof. (You may assume all standard terminology.)